Joint distribution of the process and its sojourn time for pseudo-processes governed by high-order heat equation

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Abstract

Consider the high-order heat-type equation $\partial u/\partial t = \pm\,\partial^N u/\partial x^N$ for an integer $N > 2$ and introduce the related Markov pseudo-process $(X(t))_{t\geq 0}$. In this paper, we study the sojourn time $T(t)$ in the interval $[0, +\infty)$ up to a fixed time t for this pseudo-process. We provide explicit expressions for the joint distribution of the couple $(T(t), X(t))$.

Keywords: pseudo-process, joint distribution of the process and its sojourn time, Spitzer's identity.
AMS 2000 Subject Classification: Primary 60G20; Secondary 60J25, 60K35, 60J05.

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1 Introduction

Let N be an integer equal or greater than 2 and $\kappa_{_N} = (-1)^{1+N/2}$ if N is even, $\kappa_{_N} = \pm 1$ if N is odd. Consider the heat-type equation of order N:

$$\frac{\partial u}{\partial t} = \kappa_{_N} \frac{\partial^N u}{\partial x^N}. \tag{1.1}$$

For $N = 2$, this equation is the classical normalized heat equation and its relationship with linear Brownian motion is of the most well-known. For $N > 2$, it is known that no ordinary stochastic process can be associated with this equation. Nevertheless a Markov "pseudo-process" can be constructed by imitating the case $N = 2$. This pseudo-process, $X = (X(t))_{t\geq 0}$ say, is driven by a signed measure as follows. Let $p(t;x)$ denote the elementary solution of Eq. (1.1), that is, p solves (1.1) with the initial condition $p(0;x) = \delta(x)$. This solution is characterized by its Fourier transform (see, e.g., [13])

$$\int_{-\infty}^{+\infty} e^{i\mu x}\, p(t;x)\,\mathrm{d}x = e^{\kappa_N t(-i\mu)^N}.$$

The function p is real, not always positive and its total mass is equal to one:

$$\int_{-\infty}^{+\infty} p(t;x)\,\mathrm{d}x = 1.$$

Moreover, its total absolute value mass ρ exceeds one:

$$\rho = \int_{-\infty}^{+\infty} |p(t;x)|\,\mathrm{d}x > 1.$$

In fact, if N is even, p is symmetric and $\rho < +\infty$, and if N is odd, $\rho = +\infty$. The signed function p is interpreted as the pseudo-probability for X to lie at a certain location at a certain time. More precisely, for any time $t > 0$ and any locations $x, y \in \mathbb{R}$, one defines

$$\mathbb{P}\{X(t) \in \mathrm{d}y | X(0) = x\}/\mathrm{d}y = p(t; x - y).$$

Roughly speaking, the distribution of the pseudo-process X is defined through its finite-dimensional distributions according to the Markov rule: for any $n > 1$, any times $t_1, \dots, t_n$ such that $0 < t_1 < \dots < t_n$ and any locations $x, y_1, \dots, y_n \in \mathbb{R}$,

$$\mathbb{P}\{X(t_1) \in \mathrm{d}y_1, \dots X(t_n) \in \mathrm{d}y_n | X(0) = x\}/\mathrm{d}y_1 \dots \mathrm{d}y_n = \prod_{i=1}^{n} p(t_i - t_{i-1}; y_{i-1} - y_i)$$

where $t_0 = 0$ and $y_0 = x$.

This pseudo-process has been studied by several authors: see the references [2] to [4] and the references [8] to [20].

Now, we consider the sojourn time of X in the interval $[0, +\infty)$ up to a fixed time t:

$$T(t) = \int_0^t \mathbb{1}_{[0,+\infty)}(X(s))\,\mathrm{d}s.$$

The computation of the pseudo-distribution of $T(t)$ has been done by Beghin, Hochberg, Nikitin, Orsingher and Ragozina in some particular cases (see [2, 4, 9, 16, 20]), and by Krylov and the second author in more general cases (see [10, 11]).

The method adopted therein is the use of the Feynman-Kac functional which leads to certain differential equations. We point out that the pseudo-distribution of $T(t)$ is actually a genuine probability distribution and in the case where N is even, $T(t)$ obeys the famous Paul Lévy's arcsine law, that is

$$\mathbb{P}\{T(t) \in \mathrm{ds}\}/\mathrm{ds} = \frac{\mathbb{1}_{(0,t)}(s)}{\pi\sqrt{s(t-s)}}.$$

We also mention that the sojourn time of X in a small interval $(-\varepsilon, \varepsilon)$ is used in [3] to define a local time for X at 0. The evaluation of the pseudo-distribution of the sojourn time $T(t)$ together with the up-to-date value of the pseudo-process, $X(t)$, has been tackled only in the particular cases $N = 3$ and $N = 4$

by Beghin, Hochberg, Orsingher and Ragozina (see [2, 4]). Their results have been obtained by solving certain differential equations leading to some linear systems. In [2, 4, 11], the Laplace transform of the sojourn time serves as an intermediate tool for computing the distribution of the up-to-date maximum of X.

In this paper, our aim is to derive the joint pseudo-distribution of the couple $(T(t), X(t))$ for any integer N. Since the Feynman-Kac approach used in [2, 4] leads to very cumbersome calculations, we employ an alternative method based on Spitzer's identity. Since the pseudo-process X is properly defined only in the case where N is an even integer, the results we obtain are valid in this case. Throughout the paper, we shall then assume that N is even. Nevertheless, we formally perform all computations also in the case where N is odd, even if they are not justified.

The paper is organized as follows.

- In Section 2, we write down the settings that will be used. Actually, the pseudo-process X is not well defined on the whole half-line $[0, +\infty)$. It is properly defined on dyadic times $k/2^n$, $k, n \in \mathbb{N}$. So, we introduce ad-hoc definitions for $X(t)$ and $T(t)$ as well as for some related pseudo-expectations. For instance, we shall give a meaning to the quantity

$$E(\lambda, \mu, \nu) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t + i\mu X(t) - \nu T(t)}\,\mathrm{d}t\right]$$

  which is interpreted as the 3-parameters Laplace-Fourier transform of $(T(t), X(t))$. We also recall in this part some algebraic known results.

- In Section 3, we explicitly compute $E(\lambda, \mu, \nu)$ with the help of Spitzer's identity. This is Theorem 3.1.

- Sections 4, 5 and 6 are devoted to successively inverting the Laplace-Fourier transform with respect to μ, ν and λ respectively. More precisely, in Section 4, we perform the inversion with respect to μ; this yields Theorem 4.1. Next, we perform the inversion with respect to ν which gives Theorems 5.1 and 5.2. Finally, we carry out the inversion with respect to λ and the main results of this paper are Theorems 6.2 and 6.3. In each section, we examine the particular cases $N = 2$ (case of rescaled Brownian motion), $N = 3$ (case of an asymmetric pseudo-process) and $N = 4$ (case of the biharmonic pseudo-process). Moreover, our results recover several known formulas concerning the marginal distribution of $T(t)$.

- The final appendix (Section 7) contains a discussion on Spitzer's identity as well as some technical computations.

2 Settings

2.1 A first list of settings

In this part, we introduce for each integer n a step-process X_n coinciding with the pseudo-process X on the times $k/2^n$, $k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Set, for any $k \in \mathbb{N}$, $X_{k,n} = X(k/2^n)$ and for any $t \in [k/2^n, (k+1)/2^n)$, $X(t) = X_{k,n}$. We can write globally

$$X_n(t) = \sum_{k=0}^{\infty} X_{k,n} \mathbb{1}_{[k/2^n,(k+1)/2^n)}(t).$$

Now, we recall from [13] the definitions of tame functions, functions of discrete observations, and admissible functions associated with the pseudo-process X. They were introduced by Nishioka [18] in the case $N = 4$.

Definition 2.1. *Fix $n \in \mathbb{N}$. A tame function for X is a function of a finite number k of observations of the pseudo-process X at times $j/2^n$, $1 \le j \le k$, that is a quantity of the form $F_{k,n} = F(X(1/2^n), \dots, X(k/2^n))$ for a certain k and a certain bounded Borel function $F : \mathbb{R}^k \longrightarrow \mathbb{C}$. The "expectation" of $F_{k,n}$ is defined as*

$$\mathbb{E}(F_{k,n}) = \int \dots \int_{\mathbb{R}^k} F(x_1, \dots, x_k)\, p(1/2^n; x - x_1) \dots p(1/2^n; x_{k-1} - x_k)\,\mathrm{d}x_1 \dots \mathrm{d}x_k.$$

Definition 2.2. *Fix $n \in \mathbb{N}$. A function of the discrete observations of X at times $k/2^n$, $k \geq 1$, is a convergent series of tame functions: $F_{X_n} = \sum_{k=1}^{\infty} F_{k,n}$ where $F_{k,n}$ is a tame function for all $k \geq 1$. Assuming the series $\sum_{k=1}^{\infty} |\mathbb{E}(F_{k,n})|$ convergent, the "expectation" of F_{X_n} is defined as*

$$\mathbb{E}(F_{X_n}) = \sum_{k=1}^{\infty} \mathbb{E}(F_{k,n}).$$

Definition 2.3. *An admissible function is a functional F_X of the pseudo-process X which is the limit of a sequence $(F_{X_n})_{n\in\mathbb{N}}$ of functions of discrete observations of X: $F_X = \lim_{n\to\infty} F_{X_n}$, such that the sequence $(\mathbb{E}(F_{X_n}))_{n\in\mathbb{N}}$ is convergent. The "expectation" of F_X is defined as*

$$\mathbb{E}(F_X) = \lim_{n\to\infty} \mathbb{E}(F_{X_n}).$$

In this paper, we are concerned with the sojourn time of X in $[0, +\infty)$:

$$T(t) = \int_0^t \mathbb{1}_{[0,+\infty)}(X(s))\,\mathrm{d}s.$$

In order to give a proper meaning to this quantity, we introduce the similar object related to X_n:

$$T_n(t) = \int_0^t \mathbb{1}_{[0,+\infty)}(X_n(s))\,\mathrm{d}s.$$

For determining the distribution of $T_n(t)$, we compute its 3-parameters Laplace-Fourier transform:

$$E_n(\lambda, \mu, \nu) = \mathbb{E}\left[\int_0^{\infty} e^{-\lambda t + i\mu X_n(t) - \nu T_n(t)}\,\mathrm{d}t\right].$$

In Section 3, we prove that the sequence $(E_n(\lambda, \mu, \nu))_{n\in\mathbb{N}}$ is convergent and we compute its limit:

$$\lim_{n\to\infty} E_n(\lambda, \mu, \nu) = E(\lambda, \mu, \nu).$$

Formally, $E(\lambda, \mu, \nu)$ is interpreted as

$$E(\lambda, \mu, \nu) = \mathbb{E}\left[\int_0^{\infty} e^{-\lambda t + i\mu X(t) - \nu T(t)}\,\mathrm{d}t\right]$$

where the quantity $\int_0^{\infty} e^{-\lambda t + i\mu X(t) - \nu T(t)}\,\mathrm{d}t$ is an admissible function of X. This computation is performed with the aid of Spitzer's identity. This latter concerns the classical random walk. Nevertheless, since it hinges on combinatorial arguments, it can be applied to the context of pseudo-processes. We clarify this point in Section 3.

2.2 A second list of settings

We introduce some algebraic settings. Let θ_i, $1 \leq i \leq N$, be the N^{th} roots of κ_N and

$$J = \{i \in \{1, \ldots, N\} : \Re\theta_i > 0\}, \qquad K = \{i \in \{1, \ldots, N\} : \Re\theta_i < 0\}.$$

Of course, the cardinalities of J and K sum to N: $\#J + \#K = N$. We state several results related to the θ_i's which are proved in [11, 13]. We have the elementary equalities

$$\sum_{j\in J} \theta_j + \sum_{k\in K} \theta_k = \sum_{i=1}^{N} \theta_i = 0, \qquad \left(\prod_{j\in J} \theta_j\right)\left(\prod_{k\in K} \theta_k\right) = \prod_{i=1}^{N} \theta_i = (-1)^{N-1}\kappa_N \tag{2.1}$$

and

$$\prod_{i=1}^{N}(x - \theta_i) = \prod_{i=1}^{N}(x - \bar{\theta}_i) = x^N - \kappa_N. \tag{2.2}$$

Moreover, from formula (5.10) in [13],

$$\prod_{k\in K}(x - \theta_k) = \sum_{\ell=0}^{\#K}(-1)^{\ell}\sigma_{\ell}\, x^{\#K-\ell}, \tag{2.3}$$

where $\sigma_\ell = \sum_{\substack{k_1<\dots<k_\ell\\ k_1,\dots,k_\ell\in K}} \theta_{k_1}\dots\theta_{k_\ell}$. We have by Lemma 11 in [11]

$$\sum_{j\in J}\theta_j \prod_{i\in J\setminus\{j\}} \frac{\theta_i x-\theta_j}{\theta_i-\theta_j} = \sum_{j\in J}\theta_j = -\sum_{k\in K}\theta_k = \begin{cases} \dfrac{1}{\sin\frac{\pi}{N}} & \text{if } N \text{ is even,}\\ \dfrac{1}{2\sin\frac{\pi}{2N}} = \dfrac{\cos\frac{\pi}{2N}}{\sin\frac{\pi}{N}} & \text{if } N \text{ is odd.}\end{cases} \tag{2.4}$$

Set $A_j = \prod_{i\in J\setminus\{j\}} \frac{\theta_i}{\theta_i-\theta_j}$ for $j\in J$, and $B_k = \prod_{i\in K\setminus\{k\}} \frac{\theta_i}{\theta_i-\theta_k}$ for $k\in K$. The A_j's and B_k's solve a Vandermonde system: we have

$$\begin{gathered} \sum_{j\in J} A_j = \sum_{k\in K} B_k = 1 \\ \sum_{j\in J} A_j\theta_j^m = 0 \text{ for } 1\le m\le \#J-1, \qquad \sum_{k\in K} B_k\theta_k^m = 0 \text{ for } 1\le m\le \#K-1. \end{gathered} \tag{2.5}$$

Observing that $1/\theta_j = \bar\theta_j$ for $j\in J$, that $\{\theta_j, j\in J\} = \{\bar\theta_j, j\in J\}$ and similarly for the θ_k's, $k\in K$, formula (2.11) in [13] gives

$$\begin{gathered} \sum_{j\in J}\frac{A_j\theta_j}{\theta_j-x} = \sum_{j\in J}\frac{A_j}{1-\bar\theta_j x} = \frac{1}{\prod_{j\in J}(1-\theta_j x)} = -\kappa_N \frac{\prod_{k\in K}(1-\theta_k x)}{x^N-\kappa_N} = -\kappa_N \frac{\prod_{k\in K}(1-\bar\theta_k x)}{x^N-\kappa_N}, \\ \sum_{k\in K}\frac{B_k\theta_k}{\theta_k-x} = \sum_{k\in K}\frac{B_k}{1-\bar\theta_k x} = \frac{1}{\prod_{k\in K}(1-\theta_k x)} = -\kappa_N \frac{\prod_{j\in J}(1-\theta_j x)}{x^N-\kappa_N} = -\kappa_N \frac{\prod_{j\in J}(1-\bar\theta_j x)}{x^N-\kappa_N}. \end{gathered} \tag{2.6}$$

In particular,

$$\sum_{j\in J}\frac{A_j\theta_j}{\theta_j-\theta_k} = \frac{1}{NB_k}, \qquad \sum_{k\in K}\frac{B_k\theta_k}{\theta_k-\theta_j} = \frac{1}{NA_j}. \tag{2.7}$$

Set, for any $m\in\mathbb{Z}$, $\alpha_m = \sum_{j\in J} A_j\theta_j^m$ and $\beta_m = \sum_{k\in K} B_k\theta_k^m$. We have, by formula (2.11) of [13], $\beta_{\#K} = (-1)^{\#K-1}\prod_{k\in K}\theta_k$. Moreover, $\beta_{\#K+1} = (-1)^{\#K-1}\left(\prod_{k\in K}\theta_k\right)\left(\sum_{k\in K}\theta_k\right)$. The proof of this claim is postponed to Lemma 7.2 in the appendix. We sum up this information and (2.5) into

$$\beta_m = \begin{cases} 1 & \text{if } m=0,\\ 0 & \text{if } 1\le m\le \#K-1,\\ (-1)^{\#K-1}\prod_{k\in K}\theta_k & \text{if } m=\#K,\\ (-1)^{\#K-1}\left(\prod_{k\in K}\theta_k\right)\left(\sum_{k\in K}\theta_k\right) & \text{if } m=\#K+1,\\ \kappa_N & \text{if } m=N.\end{cases} \tag{2.8}$$

We also have

$$\alpha_{-m} = \sum_{j\in J}\frac{A_j}{\theta_j^m} = \kappa_N\sum_{j\in J} A_j\theta_j^{N-m} = \kappa_N\alpha_{N-m}$$

and then

$$\alpha_{-m} = \begin{cases} 1 & \text{if } m=0,\\ \kappa_N(-1)^{\#J-1}\left(\prod_{j\in J}\theta_j\right)\left(\sum_{j\in J}\theta_j\right) & \text{if } m=\#K-1,\\ \kappa_N(-1)^{\#J-1}\prod_{j\in J}\theta_j & \text{if } m=\#K,\\ 0 & \text{if } \#K+1\le m\le N-1,\\ \kappa_N & \text{if } m=N.\end{cases} \tag{2.9}$$

In particular, by (2.1),

$$\alpha_0\beta_0 = \alpha_{-N}\beta_N = 1, \quad \alpha_{-\#K}\beta_{\#K} = -1, \quad \alpha_{-\#K}\beta_{\#K+1} = \sum_{j\in J}\theta_j, \quad \alpha_{1-\#K}\beta_{\#K} = \sum_{k\in K}\theta_k. \tag{2.10}$$

With $\sigma_0 = 1$, $\sigma_{\#K-1} = \left(\prod_{k\in K}\theta_k\right)\left(\sum_{k\in K}\bar\theta_k\right)$ and $\sigma_{\#K} = \prod_{k\in K}\theta_k$, we also have

$$\bar\sigma_0\beta_0 = 1, \quad \bar\sigma_{\#K-1}\beta_{\#K} = \bar\sigma_{\#K}\beta_{\#K+1} = (-1)^{\#K-1}\sum_{k\in K}\theta_k, \quad \bar\sigma_{\#K}\beta_{\#K} = (-1)^{\#K-1}. \tag{2.11}$$

Concerning the kernel p, we have from Proposition 1 in [11]

$$p(t;0)=\begin{cases}\dfrac{\Gamma\left(\frac{1}{N}\right)}{N\pi t^{1/N}} & \text{if } N \text{ is even,}\\[2ex] \dfrac{\Gamma\left(\frac{1}{N}\right)\cos\left(\frac{\pi}{2N}\right)}{N\pi t^{1/N}} & \text{if } N \text{ is odd.}\end{cases} \tag{2.12}$$

Proposition 3 in [11] states

$$\mathbb{P}\{X(t)\geq 0\}=\int_0^{\infty} p(t;-\xi)\,\mathrm{d}\xi=\frac{\#J}{N},\qquad \mathbb{P}\{X(t)\leq 0\}=\int_{-\infty}^{0} p(t;-\xi)\,\mathrm{d}\xi=\frac{\#K}{N} \tag{2.13}$$

and formulas (4.7) and (4.8) in [13] yield, for $\lambda>0$ and $\mu\in\mathbb{R}$,

$$\begin{aligned}\int_0^{\infty}\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\int_{-\infty}^{0}\left(e^{i\mu\xi}-1\right)p(t;-\xi)\,\mathrm{d}\xi&=\log\left(\prod_{k\in K}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-i\mu\theta_k}\right),\\ \int_0^{\infty}\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\int_{0}^{\infty}\left(e^{i\mu\xi}-1\right)p(t;-\xi)\,\mathrm{d}\xi&=\log\left(\prod_{j\in J}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-i\mu\theta_j}\right).\end{aligned} \tag{2.14}$$

Let us introduce, for $m\leq N-1$ and $x\geq 0$,

$$I_{j,m}(\tau;x)=\frac{Ni}{2\pi}\left(e^{-i\frac{m}{N}\pi}\int_0^{\infty}\xi^{N-m-1}e^{-\tau\xi^N-\theta_j e^{i\frac{\pi}{N}}x\xi}\,\mathrm{d}\xi-e^{i\frac{m}{N}\pi}\int_0^{\infty}\xi^{N-m-1}e^{-\tau\xi^N-\theta_j e^{-i\frac{\pi}{N}}x\xi}\,\mathrm{d}\xi\right). \tag{2.15}$$

Formula (5.13) in [13] gives, for $0\leq m\leq N-1$ and $x\geq 0$,

$$\int_0^{\infty}e^{-\lambda\tau}I_{j,m}(\tau;x)\,\mathrm{d}\tau=\lambda^{-\frac{m}{N}}e^{-\theta_j\sqrt[N]{\lambda}\,x}. \tag{2.16}$$

Example 2.1. Case $N=2$: we have $\kappa_2=+1$. This is the case of rescaled Brownian motion. The square roots of κ_2 are $\theta_1=1$, $\theta_2=-1$ and then $J=\{1\}$, K=$\{2\}$, $A_1=B_2=1$, $\alpha_0=\alpha_{-1}=1$, $\beta_0=1$, $\beta_{-1}=-1$. Moreover,

$$I_{1,0}(\tau;x)=\frac{i}{\pi}\left(\int_0^{\infty}\xi\,e^{-\tau\xi^2-ix\xi}\,\mathrm{d}\xi-\int_0^{\infty}\xi\,e^{-\tau\xi^2+ix\xi}\,\mathrm{d}\xi\right).$$

The function $I_{1,0}$ can be simplified. In fact, we have

$$\begin{aligned}I_{1,0}(\tau;x)&=\frac{i}{\pi}\int_{-\infty}^{\infty}\xi\,e^{-\tau\xi^2-ix\xi}\,\mathrm{d}\xi=\frac{i}{\pi}\,e^{-\frac{x^2}{4\tau}}\int_{-\infty}^{\infty}\xi\,e^{-\tau(\xi+\frac{ix}{2\tau})^2}\,\mathrm{d}\xi\\ &=\frac{i}{\pi}\,e^{-\frac{x^2}{4\tau}}\int_{-\infty}^{\infty}\left(\xi-\frac{ix}{2\tau}\right)e^{-\tau\xi^2}\,\mathrm{d}\xi=\frac{x\,e^{-\frac{x^2}{4\tau}}}{2\pi\tau}\int_{-\infty}^{\infty}e^{-\tau\xi^2}\,\mathrm{d}\xi=\frac{x\,e^{-\frac{x^2}{4\tau}}}{2\pi\tau}\int_0^{\infty}\frac{e^{-\tau\xi}}{\sqrt{\xi}}\,\mathrm{d}\xi.\end{aligned}$$

Finally,

$$I_{1,0}(\tau;x)=\frac{x\,e^{-\frac{x^2}{4\tau}}}{2\sqrt{\pi}\,\tau^{3/2}}. \tag{2.17}$$

Example 2.2. Case $N=3$.

• For $\kappa_3=+1$, the third roots of κ_3 are $\theta_1=1$, $\theta_2=e^{i\frac{2\pi}{3}}$, $\theta_3=e^{-i\frac{2\pi}{3}}$, and the settings read $J=\{1\}$, $K=\{2,3\}$, $A_1=1$, $B_2=\frac{e^{-i\frac{\pi}{6}}}{\sqrt{3}}$, $B_3=\frac{e^{i\frac{\pi}{6}}}{\sqrt{3}}$, $\alpha_0=\alpha_{-1}=\alpha_{-2}=1$, $\beta_0=1$, $\beta_{-1}=-1$. Moreover,

$$I_{1,0}(\tau;x)=\frac{3i}{2\pi}\left(\int_0^{\infty}\xi^2\,e^{-\tau\xi^3-e^{i\frac{\pi}{3}}x\xi}\,\mathrm{d}\xi-\int_0^{\infty}\xi^2\,e^{-\tau\xi^3-e^{-i\frac{\pi}{3}}x\xi}\,\mathrm{d}\xi\right).$$

• For $\kappa_3=-1$, the third roots of κ_3 are $\theta_1=e^{i\frac{\pi}{3}}$, $\theta_2=e^{-i\frac{\pi}{3}}$, $\theta_3=-1$. The settings read $J=\{1,2\}$, $K=\{3\}$, $A_1=\frac{e^{i\frac{\pi}{6}}}{\sqrt{3}}$, $A_2=\frac{e^{-i\frac{\pi}{6}}}{\sqrt{3}}$, $B_3=1$, $\alpha_0=\alpha_{-1}=1$, $\beta_0=\beta_{-2}=1$, $\beta_{-1}=-1$. Moreover,

$$\begin{aligned}I_{1,1}(\tau;x)&=\frac{3i}{2\pi}\left(e^{-i\frac{\pi}{3}}\int_0^{\infty}\xi\,e^{-\tau\xi^3-e^{i\frac{2\pi}{3}}x\xi}\,\mathrm{d}\xi-e^{i\frac{\pi}{3}}\int_0^{\infty}\xi\,e^{-\tau\xi^3-x\xi}\,\mathrm{d}\xi\right),\\ I_{2,1}(\tau;x)&=\frac{3i}{2\pi}\left(e^{-i\frac{\pi}{3}}\int_0^{\infty}\xi\,e^{-\tau\xi^3-x\xi}\,\mathrm{d}\xi-e^{i\frac{\pi}{3}}\int_0^{\infty}\xi\,e^{-\tau\xi^3-e^{-i\frac{2\pi}{3}}x\xi}\,\mathrm{d}\xi\right).\end{aligned}$$

Actually, the three functions $I_{1,0}$, $I_{1,1}$ and $I_{2,1}$ can be expressed by mean of the Airy function Hi defined as $\mathrm{Hi}(z)=\frac{1}{\pi}\int_0^\infty e^{-\frac{\xi^3}{3}+z\xi}\,\mathrm{d}\xi$ (see, e.g., [1, Chap. 10.4]). Indeed, we easily have by a change of variables, differentiation and integration by parts, for $\tau>0$ and $z\in\mathbb{C}$,

$$\int_0^\infty e^{-\tau\xi^3+z\xi}\,\mathrm{d}\xi=\frac{\pi}{(3\tau)^{4/3}}\,\mathrm{Hi}\bigg(\frac{z}{\sqrt[3]{3\tau}}\bigg),$$
$$\int_0^\infty \xi\,e^{-\tau\xi^3+z\xi}\,\mathrm{d}\xi=\frac{\pi}{(3\tau)^{2/3}}\,\mathrm{Hi}'\bigg(\frac{z}{\sqrt[3]{3\tau}}\bigg),$$
$$\int_0^\infty \xi^2\,e^{-\tau\xi^3+z\xi}\,\mathrm{d}\xi=\frac{\pi z}{(3\tau)^{4/3}}\,\mathrm{Hi}\bigg(\frac{z}{\sqrt[3]{3\tau}}\bigg)+\frac{1}{3\tau}.$$

Therefore,

$$I_{1,0}(\tau;x)=\frac{x}{2\sqrt[3]{3}\,\tau^{4/3}}\left[e^{i\frac{\pi}{6}}\,\mathrm{Hi}\bigg(-\frac{e^{-i\frac{\pi}{3}}x}{\sqrt[3]{3\tau}}\bigg)+e^{-i\frac{\pi}{6}}\,\mathrm{Hi}\bigg(-\frac{e^{i\frac{\pi}{3}}x}{\sqrt[3]{3\tau}}\bigg)\right], \tag{2.18}$$
$$I_{1,1}(\tau;x)=\frac{\sqrt[3]{3}}{2\tau^{2/3}}\left[e^{i\frac{\pi}{6}}\,\mathrm{Hi}'\bigg(-\frac{e^{i\frac{2\pi}{3}}x}{\sqrt[3]{3\tau}}\bigg)+e^{-i\frac{\pi}{6}}\,\mathrm{Hi}'\bigg(-\frac{x}{\sqrt[3]{3\tau}}\bigg)\right], \tag{2.19}$$
$$I_{2,1}(\tau;x)=\frac{\sqrt[3]{3}}{2\tau^{2/3}}\left[e^{i\frac{\pi}{6}}\,\mathrm{Hi}'\bigg(-\frac{x}{\sqrt[3]{3\tau}}\bigg)+e^{-i\frac{\pi}{6}}\,\mathrm{Hi}'\bigg(-\frac{e^{-i\frac{2\pi}{3}}x}{\sqrt[3]{3\tau}}\bigg)\right]. \tag{2.20}$$

Example 2.3. Case $N=4$: we have $\kappa_4=-1$. This is the case of the biharmonic pseudo-process. The fourth roots of κ_4 are $\theta_1=e^{-i\frac{\pi}{4}}$, $\theta_2=e^{i\frac{\pi}{4}}$, $\theta_3=e^{i\frac{3\pi}{4}}$, $\theta_4=e^{-i\frac{3\pi}{4}}$ and the notations read in this case $J=\{1,2\}$, $K=\{3,4\}$, $A_1=B_3=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}$, $A_2=B_4=\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}$, $\alpha_0=\alpha_{-2}=1$, $\alpha_{-1}=\sqrt{2}$, $\beta_0=\beta_{-2}=1$, $\beta_{-1}=-\sqrt{2}$. Moreover,

$$\begin{aligned}I_{1,1}(\tau;x)&=\frac{2}{\pi}\left(e^{i\frac{\pi}{4}}\int_0^\infty \xi^2\,e^{-\tau\xi^4-x\xi}\,\mathrm{d}\xi+e^{-i\frac{\pi}{4}}\int_0^\infty \xi^2\,e^{-\tau\xi^4+ix\xi}\,\mathrm{d}\xi\right),\\ I_{2,1}(\tau;x)&=\frac{2}{\pi}\left(e^{i\frac{\pi}{4}}\int_0^\infty \xi^2\,e^{-\tau\xi^4-ix\xi}\,\mathrm{d}\xi+e^{-i\frac{\pi}{4}}\int_0^\infty \xi^2\,e^{-\tau\xi^4-x\xi}\,\mathrm{d}\xi\right).\end{aligned} \tag{2.21}$$

3 Evaluation of $E(\lambda,\mu,\nu)$

The goal of this section is to evaluate the limit $E(\lambda,\mu,\nu)=\lim_{n\to\infty}E_n(\lambda,\mu,\nu)$. We write $E_n(\lambda,\mu,\nu)=\mathbb{E}[F_n(\lambda,\mu,\nu)]$ with

$$F_n(\lambda,\mu,\nu)=\int_0^\infty e^{-\lambda t+i\mu X_n(t)-\nu T_n(t)}\,\mathrm{d}t.$$

Let us rewrite the sojourn time $T_n(t)$ as follows:

$$\begin{aligned}T_n(t)&=\sum_{j=0}^{[2^nt]}\int_{j/2^n}^{(j+1)/2^n}\mathbb{1}_{[0,+\infty)}(X_n(s))\,\mathrm{d}s-\int_t^{([2^nt]+1)/2^n}\mathbb{1}_{[0,+\infty)}(X_n(s))\,\mathrm{d}s\\ &=\sum_{j=0}^{[2^nt]}\int_{j/2^n}^{(j+1)/2^n}\mathbb{1}_{[0,+\infty)}(X_{j,n})\,\mathrm{d}s-\int_t^{([2^nt]+1)/2^n}\mathbb{1}_{[0,+\infty)}(X_{[2^nt],n})\,\mathrm{d}s\\ &=\frac{1}{2^n}\sum_{j=0}^{[2^nt]}\mathbb{1}_{[0,+\infty)}(X_{j,n})+\left(t-\frac{[2^nt]+1}{2^n}\right)\mathbb{1}_{[0,+\infty)}(X_{[2^nt],n}).\end{aligned}$$

Set $T_{0,n}=0$ and, for $k\geq 1$,

$$T_{k,n}=\frac{1}{2^n}\sum_{j=1}^{k}\mathbb{1}_{[0,+\infty)}(X_{j,n}).$$

For $k\geq 0$ and $t\in[k/2^n,(k+1)/2^n)$, we see that

$$T_n(t)=T_{k,n}+\left(t-\frac{k+1}{2^n}\right)\mathbb{1}_{[0,+\infty)}(X_{k,n})+\frac{1}{2^n}.$$

With this decomposition at hand, we can begin to compute $F_n(\lambda,\mu,\nu)$:

$$\begin{aligned}
F_n(\lambda,\mu,\nu) &= \int_0^\infty e^{-\lambda t+i\mu X_n(t)-\nu T_n(t)}\,\mathrm{d}t\\
&= \sum_{k=0}^{\infty}\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t+i\mu X_{k,n}-\nu T_{k,n}-\frac{\nu}{2^n}+\nu(\frac{k+1}{2^n}-t)\mathbb{1}_{[0,+\infty)}(X_{k,n})}\,\mathrm{d}t\\
&= e^{-\nu/2^n}\left(\sum_{k=0}^{\infty}\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t+\nu(\frac{k+1}{2^n}-t)\mathbb{1}_{[0,+\infty)}(X_{k,n})}\,\mathrm{d}t\right)e^{i\mu X_{k,n}-\nu T_{k,n}}.
\end{aligned}$$

The value of the above integral is

$$\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t+\nu(\frac{k+1}{2^n}-t)\mathbb{1}_{[0,+\infty)}(X_{k,n})}\,\mathrm{d}t = e^{-\lambda(k+1)/2^n}\,\frac{e^{[\lambda+\nu\mathbb{1}_{[0,+\infty)}(X_{k,n})]/2^n}-1}{\lambda+\nu\mathbb{1}_{[0,+\infty)}(X_{k,n})}.$$

Therefore,

$$\begin{aligned}
F_n(\lambda,\mu,\nu) &= \frac{1-e^{-(\lambda+\nu)/2^n}}{\lambda+\nu}\sum_{k=0}^{\infty} e^{-\lambda k/2^n+i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\\
&\quad + e^{-\nu/2^n}\,\frac{1-e^{-\lambda/2^n}}{\lambda}\sum_{k=0}^{\infty} e^{-\lambda k/2^n+i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{(-\infty,0)}(X_{k,n}).
\end{aligned}$$

Before applying the expectation to this last expression, we have to check that it defines a function of discrete observations of the pseudo-process X which satisfies the conditions of Definition 2.2. This fact is stated in the proposition below.

Proposition 3.1. *Suppose N even and fix an integer n. For any complex λ such that $\Re(\lambda)>0$ and any $\nu>0$, the series $\sum_{k=0}^{\infty} e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]$ and $\sum_{k=0}^{\infty} e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{(-\infty,0)}(X_{k,n})\big]$ are absolutely convergent and their sums are given by*

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big] &= \frac{e^{\nu/2^n}-S_n^+(\lambda,\mu,\nu)}{e^{\nu/2^n}-1},\\
\sum_{k=0}^{\infty} e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{(-\infty,0)}(X_{k,n})\big] &= \frac{e^{\nu/2^n}[S_n^-(\lambda,\mu,\nu)-1]}{e^{\nu/2^n}-1},
\end{aligned}$$

where

$$\begin{aligned}
S_n^+(\lambda,\mu,\nu) &= \exp\!\left(-\sum_{k=1}^{\infty}\left(1-e^{-\nu k/2^n}\right)\frac{e^{-\lambda k/2^n}}{k}\,\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right),\\
S_n^-(\lambda,\mu,\nu) &= \exp\!\left(\sum_{k=1}^{\infty}\left(1-e^{-\nu k/2^n}\right)\frac{e^{-\lambda k/2^n}}{k}\,\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{(-\infty,0)}(X_{k,n})\big]\right).
\end{aligned}$$

PROOF

• **Step 1.** First, notice that for any $k\geq 1$, we have

$$\begin{aligned}
&\big|\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\big|\\
&\quad= \left|\int\!\cdots\!\int_{\mathbb{R}^{k-1}\times[0,+\infty)} e^{i\mu x_k-\frac{\nu}{2^n}\sum_{j=1}^k\mathbb{1}_{[0,+\infty)}(x_j)}\,\mathbb{P}\{X_{1,n}\in\mathrm{d}x_1,\dots,X_{k,n}\in\mathrm{d}x_k\}\right|\\
&\quad= \left|\int\!\cdots\!\int_{\mathbb{R}^{k-1}\times[0,+\infty)} e^{i\mu x_k-\frac{\nu}{2^n}\sum_{j=1}^k\mathbb{1}_{[0,+\infty)}(x_j)}\,p\Big(\frac{1}{2^n};x_1\Big)\prod_{j=1}^{k-1}p\Big(\frac{1}{2^n};x_j-x_{j+1}\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_k\right|\\
&\quad\leq \int\!\cdots\!\int_{\mathbb{R}^k}\left|p\Big(\frac{1}{2^n};x_1\Big)\prod_{j=1}^{k-1}p\Big(\frac{1}{2^n};x_j-x_{j+1}\Big)\right|\mathrm{d}x_1\dots\mathrm{d}x_k\\
&\quad= \int\!\cdots\!\int_{\mathbb{R}^k}\prod_{j=1}^{k}\left|p\Big(\frac{1}{2^n};y_j\Big)\right|\mathrm{d}y_1\dots\mathrm{d}y_k = \prod_{j=1}^{k}\int_{-\infty}^{+\infty}\left|p\Big(\frac{1}{2^n};y_j\Big)\right|\mathrm{d}y_j=\rho^k.
\end{aligned}$$

Hence, we derive the following inequality:

$$\sum_{k=1}^{\infty}\left|e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right| \le \sum_{k=1}^{\infty}\rho^k\left|e^{-\lambda k/2^n}\right| = \frac{1}{1-\rho e^{-\Re(\lambda)/2^n}}.$$

We can easily see that this bound holds true also when the factor $\mathbb{1}_{[0,+\infty)}(X_{k,n})$ is replaced by $\mathbb{1}_{(-\infty,0)}(X_{k,n})$. This shows that the two series of Proposition 3.1 are finite for $\lambda\in\mathbb{C}$ such that $\rho e^{-\Re(\lambda)/2^n}<1$, that is $\Re(\lambda)>2^n\log\rho$.

• **Step 2.** For $\lambda\in\mathbb{C}$ such that $\Re(\lambda)>2^n\log\rho$, the Spitzer's identity (7.2) (see Lemma 7.1 in the appendix) gives for the first series of Proposition 3.1

$$\sum_{k=0}^{\infty}e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]$$
$$=\frac{1}{e^{\nu/2^n}-1}\left[e^{\nu/2^n}-\exp\left(-\sum_{k=1}^{\infty}\left(1-e^{-\nu k/2^n}\right)\frac{e^{-\lambda k/2^n}}{k}\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right)\right]. \tag{3.1}$$

The right-hand side of (3.1) is an analytic continuation of the Dirichlet series lying in the left-hand side of (3.1), which is defined on the half-plane $\{\lambda\in\mathbb{C}:\Re(\lambda)>0\}$. Moreover, for any $\varepsilon>0$, this continuation is bounded over the half-plane $\{\lambda\in\mathbb{C}:\Re(\lambda)\ge\varepsilon\}$. Indeed, we have

$$\left|\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right| = \left|\int_0^{+\infty}e^{i\mu\xi}p\left(\frac{k}{2^n};-\xi\right)\mathrm{d}\xi\right| \le \int_0^{+\infty}\left|p\left(\frac{k}{2^n};-\xi\right)\right|\mathrm{d}\xi<\rho$$

and then

$$\left|\exp\left(-\sum_{k=1}^{\infty}\left(1-e^{-\nu k/2^n}\right)\frac{e^{-\lambda k/2^n}}{k}\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right)\right|$$
$$\le\exp\left(\rho\sum_{k=1}^{\infty}\frac{e^{-\Re(\lambda)k/2^n}}{k}\right)=\exp\Big(-\rho\log(1-e^{-\Re(\lambda)/2^n})\Big)=\frac{1}{(1-e^{-\Re(\lambda)/2^n})^{\rho}}.$$

Therefore, if $\Re(\lambda)\ge\varepsilon$,

$$\left|\exp\left(-\sum_{k=1}^{\infty}\left(1-e^{-\nu k/2^n}\right)\frac{e^{-\lambda k/2^n}}{k}\mathbb{E}\big[e^{i\mu X_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]\right)\right|\le\frac{1}{(1-e^{-\varepsilon/2^n})^{\rho}}.$$

This proves that the left-hand side of this last inequality is bounded for $\Re(\lambda)\ge\varepsilon$. By a lemma of Bohr ([5]), we deduce that the abscissas of convergence, absolute convergence and boundedness of the Dirichlet series $\sum_{k=0}^{\infty}e^{-\lambda k/2^n}\mathbb{E}\big[e^{i\mu X_{k,n}-\nu T_{k,n}}\mathbb{1}_{[0,+\infty)}(X_{k,n})\big]$ are identical. So, this series converges absolutely on the half-plane $\{\lambda\in\mathbb{C}:\Re(\lambda)>0\}$ and (3.1) holds on this half-plane. A similar conclusion holds for the second series of Proposition 3.1. The proof is finished. ■

Thanks to Proposition 3.1, we see that the functional $F_n(\lambda,\mu,\nu)$ is a function of the discrete observations of X and, by Definition 2.2, its expectation can be computed as follows:

$$\begin{aligned}E_n(\lambda,\mu,\nu)&=\frac{1-e^{-(\lambda+\nu)/2^n}}{\lambda+\nu}\,\frac{e^{\nu/2^n}-S_n^+(\lambda,\mu,\nu)}{e^{\nu/2^n}-1}+\frac{1-e^{-\lambda/2^n}}{\lambda}\,\frac{S_n^-(\lambda,\mu,\nu)-1}{e^{\nu/2^n}-1}\\&=\left(\frac{e^{\nu/2^n}(1-e^{-(\lambda+\nu)/2^n})}{(\lambda+\nu)(e^{\nu/2^n}-1)}-\frac{1-e^{-\lambda/2^n}}{\lambda(e^{\nu/2^n}-1)}\right)\\&\quad+\frac{1-e^{-\lambda/2^n}}{\lambda(e^{\nu/2^n}-1)}S_n^-(\lambda,\mu,\nu)-\frac{1-e^{-(\lambda+\nu)/2^n}}{(\lambda+\nu)(e^{\nu/2^n}-1)}S_n^+(\lambda,\mu,\nu).\end{aligned}\tag{3.2}$$

Now, we have to evaluate the limit $E(\lambda,\mu,\nu)$ of $E_n(\lambda,\mu,\nu)$ as n goes toward infinity. It is easy to see that this limit exists; see the proof of Theorem 3.1 below. Formally, we write $E(\lambda,\mu,\nu)=\mathbb{E}[F(\lambda,\mu,\nu)]$ with

$$F(\lambda,\mu,\nu)=\int_0^{\infty}e^{-\lambda t+i\mu X(t)-\nu T(t)}\,\mathrm{d}t.$$

Then, we can say that the functional $F(\lambda,\mu,\nu)$ is an admissible function of X in the sense of Definition 2.3. The value of its expectation $E(\lambda,\mu,\nu)$ is given in the following theorem.

Theorem 3.1. *The 3-parameters Laplace-Fourier transform of the couple $(T(t),X(t))$ is given by*

$$E(\lambda,\mu,\nu)=\frac{1}{\prod_{j\in J}(\sqrt[N]{\lambda+\nu}-i\mu\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}. \tag{3.3}$$

PROOF

It is plain that the term lying within the biggest parentheses in the last equality of (3.2) tends to zero as n goes towards infinity and that the coefficients lying before $S_n^+(\lambda,\mu,\nu)$ and $S_n^-(\lambda,\mu,\nu)$ tend to $1/\nu$. As a byproduct, we derive at the limit when $n\to\infty$,

$$E(\lambda,\mu,\nu)=\frac{1}{\nu}\left[S^-(\lambda,\mu,\nu)-S^+(\lambda,\mu,\nu)\right] \tag{3.4}$$

where we set

$$S^+(\lambda,\mu,\nu)=\lim_{n\to\infty}S_n^+(\lambda,\mu,\nu)=\exp\!\left(-\int_0^\infty \mathbb{E}\big[e^{i\mu X(t)}1\!\!1_{[0,+\infty)}(X(t))\big]\big(1-e^{-\nu t}\big)\,\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\right),$$
$$S^-(\lambda,\mu,\nu)=\lim_{n\to\infty}S_n^-(\lambda,\mu,\nu)=\exp\!\left(\int_0^\infty \mathbb{E}\big[e^{i\mu X(t)}1\!\!1_{(-\infty,0)}(X(t))\big]\big(1-e^{-\nu t}\big)\,\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\right).$$

We have

$$\begin{aligned}
&\int_0^\infty \mathbb{E}\big[e^{i\mu X(t)}1\!\!1_{[0,+\infty)}(X(t))\big]\big(1-e^{-\nu t}\big)\,\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\\
&\quad=\int_0^\infty \mathbb{E}\big[\big(e^{i\mu X(t)}-1\big)1\!\!1_{[0,+\infty)}(X(t))\big]\,\frac{e^{-\lambda t}}{t}\,\mathrm{d}t-\int_0^\infty \mathbb{E}\big[\big(e^{i\mu X(t)}-1\big)1\!\!1_{[0,+\infty)}(X(t))\big]\,\frac{e^{-(\lambda+\nu)t}}{t}\,\mathrm{d}t\\
&\qquad+\int_0^\infty \mathbb{P}\{X(t)\ge 0\}\,\frac{e^{-\lambda t}-e^{-(\lambda+\nu)t}}{t}\,\mathrm{d}t\\
&\quad=\int_0^\infty \frac{e^{-\lambda t}}{t}\,\mathrm{d}t\int_0^\infty \big(e^{i\mu\xi}-1\big)\,p(t;-\xi)\,\mathrm{d}\xi-\int_0^\infty \frac{e^{-(\lambda+\nu)t}}{t}\,\mathrm{d}t\int_0^\infty \big(e^{i\mu\xi}-1\big)\,p(t;-\xi)\,\mathrm{d}\xi\\
&\qquad+\mathbb{P}\{X(1)\ge 0\}\int_0^\infty \frac{e^{-\lambda t}-e^{-(\lambda+\nu)t}}{t}\,\mathrm{d}t.
\end{aligned}$$

In view of (2.13) and (2.14) and using the elementary equality $\int_0^\infty \frac{e^{-\lambda t}-e^{-(\lambda+\nu)t}}{t}\,\mathrm{d}t=\log\left(\frac{\lambda+\nu}{\lambda}\right)$, we have

$$\begin{aligned}
&\int_0^\infty \mathbb{E}\big[e^{i\mu X(t)}1\!\!1_{[0,+\infty)}(X(t))\big]\big(1-e^{-\nu t}\big)\,\frac{e^{-\lambda t}}{t}\,\mathrm{d}t\\
&\quad=\log\!\left(\prod_{j\in J}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-i\mu\theta_j}\right)-\log\!\left(\prod_{j\in J}\frac{\sqrt[N]{\lambda+\nu}}{\sqrt[N]{\lambda+\nu}-i\mu\theta_j}\right)+\frac{\#J}{N}\log\!\left(\frac{\lambda+\nu}{\lambda}\right)=\log\!\left(\prod_{j\in J}\frac{\sqrt[N]{\lambda+\nu}-i\mu\theta_j}{\sqrt[N]{\lambda}-i\mu\theta_j}\right).
\end{aligned}$$

We then deduce the value of $S^+(\lambda,\mu,\nu)$. By (2.2),

$$\begin{aligned}
S^+(\lambda,\mu,\nu)&=\prod_{j\in J}\frac{\sqrt[N]{\lambda}-i\mu\theta_j}{\sqrt[N]{\lambda+\nu}-i\mu\theta_j}=\frac{\prod_{\ell=1}^N(\sqrt[N]{\lambda}-i\mu\theta_\ell)}{\prod_{j\in J}(\sqrt[N]{\lambda+\nu}-i\mu\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}\\
&=\frac{\lambda-\kappa_N(i\mu)^N}{\prod_{j\in J}(\sqrt[N]{\lambda+\nu}-i\mu\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}.
\end{aligned} \tag{3.5}$$

Similarly, the value of $S^-(\lambda,\mu,\nu)$ is given by

$$S^-(\lambda,\mu,\nu)=\prod_{k\in K}\frac{\sqrt[N]{\lambda+\nu}-i\mu\theta_k}{\sqrt[N]{\lambda}-i\mu\theta_k}=\frac{\lambda+\nu-\kappa_N(i\mu)^N}{\prod_{j\in J}(\sqrt[N]{\lambda+\nu}-i\mu\theta_j)\prod_{k\in K}(\sqrt[N]{\lambda}-i\mu\theta_k)}. \tag{3.6}$$

Finally, putting (3.5) and (3.6) into (3.4) immediately leads to (3.3). ■

Remark 3.1. In the particular case $\mu=0$, we get the very simple result:

$$E(\lambda,0,\nu)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big[e^{-\nu T(t)}\big]\,\mathrm{d}t=\frac{1}{\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}}.$$

This is formula (20) of [11]. On the other hand, we can rewrite (3.3) as

$$E(\lambda,\mu,\nu)=\frac{1}{\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}}\prod_{j\in J}\frac{\sqrt[N]{\lambda+\nu}}{\sqrt[N]{\lambda+\nu}-i\mu\theta_j}\prod_{k\in K}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-i\mu\theta_k}. \tag{3.7}$$

Actually, this form is more suitable for the inversion of the Laplace-Fourier transform.

In the three next sections, we progressively invert the 3-parameters Laplace-Fourier transform $E(\lambda,\mu,\nu)$.

4 Inverting with respect to μ

In this part, we invert $E(\lambda,\mu,\nu)$ given by (3.7) with respect to μ.

Theorem 4.1. *We have, for $\lambda,\nu>0$,*

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\in\mathrm{d}x\big)/\mathrm{d}x\right]\mathrm{d}t\\
&\qquad=\begin{cases}\dfrac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\displaystyle\sum_{j\in J}A_j\theta_j\left(\sum_{k\in K}\frac{B_k\theta_k}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right)e^{-\theta_j\sqrt[N]{\lambda+\nu}\,x} & \text{if } x\ge 0,\\[2ex]
\dfrac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\displaystyle\sum_{k\in K}B_k\theta_k\left(\sum_{j\in J}\frac{A_j\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right)e^{-\theta_k\sqrt[N]{\lambda}\,x} & \text{if } x\le 0.\end{cases}
\end{aligned}\tag{4.1}$$

Proof
By (2.6) applied to $x=i\mu/\sqrt[N]{\lambda+\nu}$ and $x=i\mu/\sqrt[N]{\lambda}$, we have

$$\begin{aligned}
\prod_{j\in J}\frac{\sqrt[N]{\lambda+\nu}}{\sqrt[N]{\lambda+\nu}-i\mu\theta_j}\prod_{k\in K}\frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda}-i\mu\theta_k}&=\prod_{j\in J}\frac{1}{1-\frac{i\mu}{\sqrt[N]{\lambda+\nu}}\,\theta_j}\prod_{k\in K}\frac{1}{1-\frac{i\mu}{\sqrt[N]{\lambda}}\,\theta_k}\\
&=\sum_{j\in J}\frac{A_j\theta_j}{\theta_j-\frac{i\mu}{\sqrt[N]{\lambda+\nu}}}\sum_{k\in K}\frac{B_k\theta_k}{\theta_k-\frac{i\mu}{\sqrt[N]{\lambda}}}\\
&=\sqrt[N]{\lambda(\lambda+\nu)}\sum_{\substack{j\in J\\k\in K}}\frac{A_jB_k\theta_j\theta_k}{(\theta_j\sqrt[N]{\lambda+\nu}-i\mu)(\theta_k\sqrt[N]{\lambda}-i\mu)}.
\end{aligned}$$

Let us write that

$$\begin{aligned}
\frac{1}{(\theta_j\sqrt[N]{\lambda+\nu}-i\mu)(\theta_k\sqrt[N]{\lambda}-i\mu)}&=\frac{1}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\mu}}\left(\frac{1}{\theta_j\sqrt[N]{\lambda+\nu}-i\mu}-\frac{1}{\theta_k\sqrt[N]{\lambda}-i\mu}\right)\\
&=\frac{1}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\mu}}\left(\int_0^\infty e^{(i\mu-\theta_j\sqrt[N]{\lambda+\mu})x}\,\mathrm{d}x+\int_{-\infty}^0 e^{(i\mu-\theta_k\sqrt[N]{\lambda})x}\,\mathrm{d}x\right).
\end{aligned}$$

Therefore, we can rewrite $E(\lambda,\mu,\nu)$ as

$$\begin{aligned}
E(\lambda,\mu,\nu)=&\frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\\
&\times\sum_{\substack{j\in J\\k\in K}}\frac{A_jB_k\theta_j\theta_k}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\int_{-\infty}^\infty e^{i\mu x}\left(e^{-\theta_k\sqrt[N]{\lambda}\,x}\mathbb{1}_{(-\infty,0]}(x)+e^{-\theta_j\sqrt[N]{\lambda+\nu}\,x}\mathbb{1}_{[0,\infty)}(x)\right)\mathrm{d}x
\end{aligned}$$

which is nothing but the Fourier transform with respect to μ of the right-hand side of (4.1). ■

Remark 4.1. • By integrating (4.1) on $(-\infty,0]$, we obtain

$$\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\le 0\big)\,\mathrm{d}t=-\frac{1}{\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{k\in K}B_k\left(\sum_{j\in J}\frac{A_j\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right).$$

Using (2.6) applied to $x=\theta_k\sqrt[N]{\lambda}/\sqrt[N]{\lambda+\nu}$ and (2.3), we see that

$$
\begin{aligned}
\sum_{j\in J}\frac{A_j\theta_j}{\theta_j-\theta_k\sqrt[N]{\frac{\lambda}{\lambda+\nu}}} &= \frac{\lambda+\nu}{\nu}\prod_{i\in K}\left(1-\bar\theta_i\theta_k\sqrt[N]{\frac{\lambda}{\lambda+\nu}}\right)\\
&= \theta_k^{\#K}\,\frac{\lambda+\nu}{\nu}\left(\frac{\lambda}{\lambda+\nu}\right)^{\frac{\#K}{N}}\prod_{i\in K}\left(\bar\theta_k\sqrt[N]{\frac{\lambda+\nu}{\lambda}}-\bar\theta_i\right)\\
&= \frac{1}{\nu}\,\theta_k^{\#K}\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}\sum_{\ell=0}^{\#K}(-1)^\ell\bar\sigma_\ell\,\bar\theta_k^{\#K-\ell}\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}\\
&= \frac{1}{\nu}\,\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}\sum_{\ell=0}^{\#K}(-1)^\ell\bar\sigma_\ell\,\theta_k^{\ell}\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}.
\end{aligned}
\tag{4.2}
$$

This entails that

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\le 0\big)\,\mathrm{d}t &= \frac{1}{\nu}\sum_{k\in K}B_k\sum_{\ell=0}^{\#K}(-1)^\ell\bar\sigma_\ell\,\theta_k^\ell\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}\\
&= \frac{1}{\nu}\sum_{\ell=0}^{\#K}(-1)^\ell\bar\sigma_\ell\,\beta_\ell\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}.
\end{aligned}
$$

By (2.8), we know that all the β_ℓ, $1\le\ell\le\#K-1$ vanish and it remains, with (2.11),

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\le 0\big)\,\mathrm{d}t &= \frac{1}{\nu}\left[\bar\sigma_0\beta_0\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K}{N}}+(-1)^{\#K}\bar\sigma_{\#K}\beta_{\#K}\right]\\
&= \frac{1}{\nu}\left[\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K}{N}}-1\right].
\end{aligned}
\tag{4.3}
$$

We retrieve (30) of [11].

• Likewise, we have

$$
\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\le 0\big)\,\mathrm{d}t = \frac{1}{\nu}\left[1-\left(\frac{\lambda}{\lambda+\nu}\right)^{\frac{\#J}{N}}\right],
\tag{4.4}
$$

which coincides with (29) of [11].

• Adding formulas (4.3) and (4.4) we obtain

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)}\big)\,\mathrm{d}t &= \frac{1}{\nu}\left[\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K}{N}}-\left(\frac{\lambda}{\lambda+\nu}\right)^{\frac{\#J}{N}}\right] = \frac{(\lambda+\nu)^{\frac{\#J+\#K}{N}}-\lambda^{\frac{\#J+\#K}{N}}}{\nu\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}}\\
&= \frac{1}{\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}}.
\end{aligned}
$$

This is formula (10) of [11] which has already been pointed out in Remark 3.1. Another way of checking this formula consists of integrating (4.1) with respect to x directly on $\mathbb{R}$. Indeed,

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)}\big)\,\mathrm{d}t\\
&\quad= \frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\left(\frac{1}{\sqrt[N]{\lambda+\nu}}\sum_{\substack{j\in J\\k\in K}}\frac{A_jB_k\theta_k}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}-\frac{1}{\sqrt[N]{\lambda}}\sum_{\substack{j\in J\\k\in K}}\frac{A_jB_k\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right)\\
&\quad= \frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{\substack{j\in J\\k\in K}}\left(\frac{A_jB_k\theta_k}{\sqrt[N]{\lambda+\nu}\,(\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu})}-\frac{A_jB_k\theta_j}{\sqrt[N]{\lambda}\,(\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu})}\right)\\
&\quad= \frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{\substack{j\in J\\k\in K}}\frac{A_jB_k}{\sqrt[N]{\lambda(\lambda+\nu)}}.
\end{aligned}
$$

By (2.5), we have $\sum_{\substack{j\in J\\k\in K}} A_jB_k = \sum_{j\in J} A_j \sum_{k\in K} B_k = 1$ and then

$$\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)}\big)\,\mathrm{d}t = \frac{1}{\lambda^{\frac{\#K}{N}}(\lambda+\nu)^{\frac{\#J}{N}}}.$$

Remark 4.2. By replacing x by 0 into (4.1) and by using (4.2), we get

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\in\mathrm{d}x\big)/\mathrm{d}x\Big|_{x=0}\mathrm{d}t &= \frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{k\in K} B_k\theta_k\left(\sum_{j\in J}\frac{A_j\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right).\\
&= -\frac{\sqrt[N]{\lambda}}{\nu}\sum_{k\in K} B_k\theta_k\sum_{\ell=0}^{\#K}(-1)^\ell\bar{\sigma}_\ell\,\theta_k^\ell\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}\\
&= -\frac{\sqrt[N]{\lambda}}{\nu}\sum_{\ell=0}^{\#K}(-1)^\ell\bar{\sigma}_\ell\,\beta_{\ell+1}\left(\frac{\lambda+\nu}{\lambda}\right)^{\frac{\#K-\ell}{N}}.\end{aligned}$$

In this sum, all the $\beta_{\ell+1}$, $0\le\ell\le\#K-2$, vanish and it remains, with (2.11),

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\,\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\in\mathrm{d}x\big)/\mathrm{d}x\Big|_{x=0}\mathrm{d}t\\
&\quad= -\frac{\sqrt[N]{\lambda}}{\nu}\left((-1)^{\#K-1}\bar{\sigma}_{\#K-1}\beta_{\#K}\sqrt[N]{\frac{\lambda+\nu}{\lambda}}+(-1)^{\#K}\bar{\sigma}_{\#K}\beta_{\#K+1}\right)\\
&\quad= -\frac{\sqrt[N]{\lambda}}{\nu}\sum_{k\in K}\theta_k\left(\sqrt[N]{\frac{\lambda+\nu}{\lambda}}-1\right) = \left(\sum_{j\in J}\theta_j\right)\frac{\sqrt[N]{\lambda+\nu}-\sqrt[N]{\lambda}}{\nu}.\end{aligned}$$

We retrieve formula (26) of [11].

Remark 4.3. For $\nu=0$, formula (4.1) yields with (2.7) the λ-potential of X:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{X(t)\in\mathrm{d}x\}/\mathrm{d}x]\,\mathrm{d}t &= \begin{cases}\dfrac{1}{\lambda^{1-\frac{1}{N}}}\displaystyle\sum_{j\in J}A_j\theta_j\sum_{k\in K}\frac{B_k\theta_k}{\theta_k-\theta_j}\,e^{-\theta_j\sqrt[N]{\lambda}\,x} & \text{if } x\ge 0,\\ \dfrac{1}{\lambda^{1-\frac{1}{N}}}\displaystyle\sum_{k\in K}B_k\theta_k\sum_{j\in J}\frac{A_j\theta_j}{\theta_k-\theta_j}\,e^{-\theta_k\sqrt[N]{\lambda}\,x} & \text{if } x\le 0,\end{cases}\\
&= \begin{cases}\dfrac{1}{N\lambda^{1-\frac{1}{N}}}\displaystyle\sum_{j\in J}\theta_j\,e^{-\theta_j\sqrt[N]{\lambda}\,x} & \text{if } x\ge 0,\\ -\dfrac{1}{N\lambda^{1-\frac{1}{N}}}\displaystyle\sum_{k\in K}\theta_k e^{-\theta_k\sqrt[N]{\lambda}\,x} & \text{if } x\le 0.\end{cases}\end{aligned}$$

We retrieve (12) of [11].

Example 4.1. For $N=2$, formula (4.1) gives, with the numerical values of Example 2.1,

$$\int_0^\infty e^{-\lambda t}\left[\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\in\mathrm{d}x\big)/\mathrm{d}x\right]\mathrm{d}t = \begin{cases}\dfrac{1}{\sqrt{\lambda}+\sqrt{\lambda+\nu}}\,e^{-\sqrt{\lambda+\nu}\,x} & \text{if } x\ge 0,\\ \dfrac{1}{\sqrt{\lambda}+\sqrt{\lambda+\nu}}\,e^{\sqrt{\lambda}\,x} & \text{if } x\le 0.\end{cases}$$

This is formula 1.4.5, p. 129, of [6].

Example 4.2. For $N=3$, we have two cases to consider. Formula (4.1) yields, with the numerical values of Example 2.2, in the case $\kappa_3=1$,

$$\int_0^{\infty} e^{-\lambda t} \left[\mathbb{E}\big(e^{-\nu T(t)},\, X(t)\in \mathrm{d}x\big)/\mathrm{d}x\right] \mathrm{d}t$$
$$= \begin{cases} \dfrac{e^{-\sqrt[3]{\lambda+\nu}\,x}}{\lambda^{2/3}+\sqrt[3]{\lambda(\lambda+\nu)}+(\lambda+\nu)^{2/3}} & \text{if } x\geq 0,\\[2ex] \dfrac{e^{\frac{\sqrt[3]{\lambda}}{2}x}}{\sqrt{3}\,\sqrt[3]{\lambda}}\,\dfrac{\sqrt{3}\,\sqrt[3]{\lambda}\,\cos\!\big(\frac{\sqrt{3}\sqrt[3]{\lambda}}{2}\,x\big)-\big(2\sqrt[3]{\lambda+\nu}+\sqrt[3]{\lambda}\,\big)\sin\!\big(\frac{\sqrt{3}\sqrt[3]{\lambda}}{2}\,x\big)}{\lambda^{2/3}+\sqrt[3]{\lambda(\lambda+\nu)}+(\lambda+\nu)^{2/3}} & \text{if } x\leq 0, \end{cases}$$

and in the case $\kappa_3=-1$,

$$\int_0^{\infty} e^{-\lambda t} \left[\mathbb{E}\big(e^{-\nu T(t)},\, X(t)\in \mathrm{d}x\big)/\mathrm{d}x\right] \mathrm{d}t$$
$$= \begin{cases} \dfrac{e^{-\frac{\sqrt[3]{\lambda+\nu}}{2}x}}{\sqrt{3}\sqrt[3]{\lambda+\nu}}\,\dfrac{\sqrt{3}\,\sqrt[3]{\lambda+\nu}\,\cos\!\big(\frac{\sqrt{3}\sqrt[3]{\lambda+\nu}}{2}\,x\big)+\big(\sqrt[3]{\lambda+\nu}+2\sqrt[3]{\lambda}\,\big)\sin\!\big(\frac{\sqrt{3}\sqrt[3]{\lambda+\nu}}{2}\,x\big)}{\lambda^{2/3}+\sqrt[3]{\lambda(\lambda+\nu)}+(\lambda+\nu)^{2/3}} & \text{if } x\geq 0,\\[2ex] \dfrac{e^{\sqrt[3]{\lambda}\,x}}{\lambda^{2/3}+\sqrt[3]{\lambda(\lambda+\nu)}+(\lambda+\nu)^{2/3}} & \text{if } x\leq 0. \end{cases}$$

Example 4.3. For $N=4$, formula (4.1) supplies, with the numerical values of Example 2.3,

$$\int_0^{\infty} e^{-\lambda t} \left[\mathbb{E}\big(e^{-\nu T(t)},\, X(t)\in \mathrm{d}x\big)/\mathrm{d}x\right] \mathrm{d}t$$
$$= \begin{cases} \dfrac{\sqrt{2}\,e^{-\frac{\sqrt[4]{\lambda+\nu}}{\sqrt{2}}x}}{\sqrt[4]{\lambda+\nu}\,(\sqrt{\lambda}+\sqrt{\lambda+\nu})(\sqrt[4]{\lambda}+\sqrt[4]{\lambda+\nu})}\left[\sqrt[4]{\lambda+\nu}\,\cos\!\left(\dfrac{\sqrt[4]{\lambda+\nu}}{\sqrt{2}}\,x\right)+\sqrt[4]{\lambda}\,\sin\!\left(\dfrac{\sqrt[4]{\lambda+\nu}}{\sqrt{2}}\,x\right)\right] & \text{if } x\geq 0,\\[2ex] \dfrac{\sqrt{2}\,e^{\frac{\sqrt[4]{\lambda}}{\sqrt{2}}x}}{\sqrt[4]{\lambda}\,(\sqrt{\lambda}+\sqrt{\lambda+\nu})(\sqrt[4]{\lambda}+\sqrt[4]{\lambda+\nu})}\left[\sqrt[4]{\lambda}\,\cos\!\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}\,x\right)-\sqrt[4]{\lambda+\nu}\,\sin\!\left(\dfrac{\sqrt[4]{\lambda}}{\sqrt{2}}\,x\right)\right] & \text{if } x\leq 0. \end{cases}$$

Remark 4.4. Using quite analogous computations to those of Remark 4.2, we could derive another expression for formula (4.1). Actually it will not be used for the inversion.

$$\int_0^{\infty} e^{-\lambda t} \left[\mathbb{E}\big(e^{-\nu T(t)},\, X(t)\in \mathrm{d}x\big)/\mathrm{d}x\right] \mathrm{d}t$$
$$= \begin{cases} \dfrac{\sqrt[N]{\lambda+\nu}-\sqrt[N]{\lambda}}{\nu}\displaystyle\sum_{j\in J}\theta_j\left(\prod_{i\in J\setminus\{j\}}\frac{\theta_i\sqrt[N]{\frac{\lambda}{\lambda+\nu}}-\theta_j}{\theta_i-\theta_j}\right)e^{-\theta_j\sqrt[N]{\lambda+\nu}\,x} & \text{if } x\geq 0,\\[2ex] -\dfrac{\sqrt[N]{\lambda+\nu}-\sqrt[N]{\lambda}}{\nu}\displaystyle\sum_{k\in K}\theta_k\left(\prod_{i\in K\setminus\{k\}}\frac{\theta_i\sqrt[N]{\frac{\lambda+\nu}{\lambda}}-\theta_k}{\theta_i-\theta_k}\right)e^{-\theta_k\sqrt[N]{\lambda}\,x} & \text{if } x\leq 0. \end{cases} \tag{4.5}$$

Formula (4.5) looks like formula (24) in [11]. Nevertheless, (4.5) involves the distribution of $(T(t),X(t))$ when the pseudo-process starts at zero while (24) of [11] involves the density of $(T(t),X(t))$ evaluated at the extremity $X(t)=0$ when the starting point is x. Actually, both formulas are identical by invoking the duality upon changing x into $-x$, but they were obtained through different approaches.

5 Inverting with respect to ν

In this section, we carry out the inversion with respect to the parameter ν. The cases $x\leq 0$ and $x\geq 0$ lead to results which are not quite analogous. This is due to the asymmetry of our problem. So, we split our analysis into two subsections related to the cases $x\leq 0$ and $x\geq 0$.

5.1 The case $x\leq 0$

Theorem 5.1. *The Laplace transform with respect to t of the density of the couple $(T(t),X(t))$ is given, when $x\leq 0$, by*

$$\int_0^\infty e^{-\lambda t}\left[\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in\mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\right]\mathrm{d}t$$
$$= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}}\sum_{m=0}^{\#K}\alpha_{-m}\,(\lambda s)^{\frac{m}{N}} E_{1,\frac{m+\#J}{N}}(\lambda s)\sum_{k\in K} B_k\theta_k^{m+1} e^{-\theta_k\sqrt[N]{\lambda}\,x}. \tag{5.1}$$

PROOF

Recall (4.1) in the case $x\le 0$:

$$\int_0^\infty e^{-\lambda t}\left[\mathbb{E}\big(e^{-\nu T(t)},\, X(t)\in\mathrm{d}x\big)/\mathrm{d}x\right]\mathrm{d}t$$
$$= \frac{1}{\lambda^{\frac{\#K-1}{N}}(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{k\in K} B_k\theta_k\left(\sum_{j\in J}\frac{A_j\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}}\right)e^{-\theta_k\sqrt[N]{\lambda}\,x}.$$

We have to invert with respect to ν the quantity

$$\frac{1}{(\lambda+\nu)^{\frac{\#J-1}{N}}}\sum_{j\in J}\frac{A_j\theta_j}{\theta_k\sqrt[N]{\lambda}-\theta_j\sqrt[N]{\lambda+\nu}} = -\sum_{j\in J}\frac{A_j}{(\lambda+\nu)^{\frac{\#J-1}{N}}\left(\sqrt[N]{\lambda+\nu}-\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda}\right)}.$$

By using the following elementary equality, which is valid for $\alpha>0$,

$$\frac{1}{(\lambda+\nu)^\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-(\lambda+\nu)s}s^{\alpha-1}\,\mathrm{d}s=\int_0^\infty e^{-\nu s}\left(\frac{s^{\alpha-1}e^{-\lambda s}}{\Gamma(\alpha)}\right)\mathrm{d}s,$$

we obtain, for $|\beta|<\sqrt[N]{\lambda+\nu}$,

$$\frac{1}{\sqrt[N]{\lambda+\nu}-\beta}=\frac{1}{\sqrt[N]{\lambda+\nu}}\,\frac{1}{1-\frac{\beta}{\sqrt[N]{\lambda+\nu}}}=\sum_{r=0}^\infty\frac{\beta^r}{(\lambda+\nu)^{\frac{r+1}{N}}}=\sum_{r=0}^\infty\frac{\beta^r}{\Gamma\big(\frac{r+1}{N}\big)}\int_0^\infty e^{-(\lambda+\nu)s}s^{\frac{r+1}{N}-1}\,\mathrm{d}s$$
$$=\int_0^\infty e^{-\nu s}\left(s^{\frac{1}{N}-1}e^{-\lambda s}\sum_{r=0}^\infty\frac{(\beta\sqrt[N]{s}\,)^r}{\Gamma\big(\frac{r+1}{N}\big)}\right)\mathrm{d}s.$$

The sum lying in the last displayed equality can be expressed by means of the Mittag-Leffler function (see [7, Chap. XVIII]): $E_{a,b}(\xi)=\sum_{r=0}^\infty\frac{\xi^r}{\Gamma(ar+b)}$. Then,

$$\frac{1}{\sqrt[N]{\lambda+\nu}-\beta}=\int_0^\infty e^{-\nu s}\left(s^{\frac{1}{N}-1}e^{-\lambda s}E_{\frac{1}{N},\frac{1}{N}}(\beta\sqrt[N]{s}\,)\right)\mathrm{d}s. \tag{5.2}$$

Next, we write

$$\sum_{j\in J}\frac{A_j}{\sqrt[N]{\lambda+\nu}-\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda}}=\int_0^\infty e^{-\nu s}\left[s^{\frac{1}{N}-1}e^{-\lambda s}\sum_{j\in J}A_j\,E_{\frac{1}{N},\frac{1}{N}}\left(\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda s}\right)\right]\mathrm{d}s, \tag{5.3}$$

where

$$\sum_{j\in J}A_j\,E_{\frac{1}{N},\frac{1}{N}}\left(\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda s}\right)=\sum_{j\in J}A_j\sum_{r=0}^\infty\left(\frac{\theta_k}{\theta_j}\right)^r\frac{(\lambda s)^{\frac{r}{N}}}{\Gamma\big(\frac{r+1}{N}\big)}=\sum_{r=0}^\infty\left(\theta_k^r\sum_{j\in J}\frac{A_j}{\theta_j^r}\right)\frac{(\lambda s)^{\frac{r}{N}}}{\Gamma\big(\frac{r+1}{N}\big)}.$$

When performing the euclidian division of r by N, we can write r as $r=\ell N+m$ with $\ell\ge 0$ and $0\le m\le N-1$. With this, we have $\theta_j^{-r}=(\theta_j^N)^{-\ell}\,\theta_j^{-m}=\kappa_N^\ell\,\theta_j^{-m}$ and $\theta_k^r=\kappa_N^\ell\,\theta_k^m$. Then,

$$\theta_k^r\sum_{j\in J}\frac{A_j}{\theta_j^r}=\theta_k^m\sum_{j\in J}\frac{A_j}{\theta_j^m}=\theta_k^m\alpha_{-m}.$$

Hence, since by (2.9) the α_{-m}, $\#K+1\le m\le N$, vanish,

$$\sum_{j\in J}A_j\,E_{\frac{1}{N},\frac{1}{N}}\left(\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda s}\right)=\sum_{\ell=0}^\infty\sum_{m=0}^{\#K}\alpha_{-m}\theta_k^m\frac{(\lambda s)^{\ell+\frac{m}{N}}}{\Gamma\big(\ell+\frac{m+1}{N}\big)}=\sum_{m=0}^{\#K}\alpha_{-m}\,\theta_k^m(\lambda s)^{\frac{m}{N}}E_{1,\frac{m+1}{N}}(\lambda s)$$

and (5.3) becomes

$$\sum_{j\in J}\frac{A_j}{\sqrt[N]{\lambda+\nu}-\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda}}=\int_0^\infty e^{-\nu s}\left(s^{\frac{1}{N}-1}e^{-\lambda s}\sum_{m=0}^{\#K}\alpha_{-m}\,\theta_k^m(\lambda s)^{\frac{m}{N}}E_{1,\frac{m+1}{N}}(\lambda s)\right)\mathrm{d}s.$$

As a result, by introducing a convolution product, we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[\mathbb{E}\big(e^{-\nu T(t)},\,X(t)\in\mathrm{d}x\big)/\mathrm{d}x\right]\mathrm{d}t\\
&\quad=-\frac{1}{\lambda^{\frac{\#K-1}{N}}}\sum_{k\in K}B_k\theta_k e^{-\theta_k\sqrt[N]{\lambda}\,x}\\
&\qquad\times\int_0^\infty e^{-\nu s}\left(\int_0^s\frac{\sigma^{\frac{\#J-1}{N}-1}e^{-\lambda\sigma}}{\Gamma\big(\frac{\#J-1}{N}\big)}\times e^{-\lambda(s-\sigma)}\sum_{m=0}^{\#K}\alpha_{-m}\theta_k^m\lambda^{\frac{m}{N}}(s-\sigma)^{\frac{m+1}{N}-1}E_{1,\frac{m+1}{N}}(\lambda(s-\sigma))\,\mathrm{d}\sigma\right)\mathrm{d}s.
\end{aligned}$$

By removing the Laplace transforms with respect to the parameter ν of each member of the foregoing equality, we extract

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[\mathbb{P}\{T(t)\in\mathrm{d}s,\,X(t)\in\mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\right]\mathrm{d}t\\
&\quad=-\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}}}\sum_{m=0}^{\#K}\alpha_{-m}\lambda^{\frac{m}{N}}\left(\sum_{k\in K}B_k\theta_k^{m+1}e^{-\theta_k\sqrt[N]{\lambda}\,x}\right)\int_0^s\frac{\sigma^{\frac{\#J-1}{N}-1}}{\Gamma\big(\frac{\#J-1}{N}\big)}(s-\sigma)^{\frac{m+1}{N}-1}E_{1,\frac{m+1}{N}}(\lambda(s-\sigma))\,\mathrm{d}\sigma.
\end{aligned}$$

The integral lying on the right-hand side of the previous equality can be evaluated as follows:

$$\begin{aligned}
\int_0^s\frac{\sigma^{\frac{\#J-1}{N}-1}}{\Gamma\big(\frac{\#J-1}{N}\big)}(s-\sigma)^{\frac{m+1}{N}-1}E_{1,\frac{m+1}{N}}(\lambda(s-\sigma))\,\mathrm{d}\sigma&=\int_0^s\frac{\sigma^{\frac{\#J-1}{N}-1}}{\Gamma\big(\frac{\#J-1}{N}\big)}(s-\sigma)^{\frac{m+1}{N}-1}\sum_{\ell=0}^\infty\frac{\lambda^\ell(s-\sigma)^\ell}{\Gamma\big(\ell+\frac{m+1}{N}\big)}\,\mathrm{d}\sigma\\
&=\sum_{\ell=0}^\infty\lambda^\ell\int_0^s\frac{\sigma^{\frac{\#J-1}{N}-1}}{\Gamma\big(\frac{\#J-1}{N}\big)}\frac{(s-\sigma)^{\ell+\frac{m+1}{N}-1}}{\Gamma\big(\ell+\frac{m+1}{N}\big)}\,\mathrm{d}\sigma\\
&=\sum_{\ell=0}^\infty\frac{\lambda^\ell s^{\ell+\frac{m+\#J}{N}-1}}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)}=s^{\frac{m+\#J}{N}-1}E_{1,\frac{m+\#J}{N}}(\lambda s)
\end{aligned}$$

from which we deduce (5.1). ■

Remark 5.1. Let us integrate formula (5.1) with respect to x on $(-\infty,0]$. This gives

$$\int_0^\infty e^{-\lambda t}\left[\mathbb{P}\{T(t)\in\mathrm{d}s,\,X(t)\le 0\}/\mathrm{d}s\right]\mathrm{d}t=\frac{e^{-\lambda s}}{(\lambda s)^{\frac{\#K}{N}}}\sum_{m=0}^{\#K}\alpha_{-m}\beta_m(\lambda s)^{\frac{m}{N}}E_{1,\frac{m+\#J}{N}}(\lambda s).$$

In view of (2.9) and (2.10), since $E_{1,1}(\lambda s)=e^{\lambda s}$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t}\left[\mathbb{P}\{T(t)\in\mathrm{d}s,\,X(t)\le 0\}/\mathrm{d}s\right]\mathrm{d}t&=\frac{e^{-\lambda s}}{(\lambda s)^{\frac{\#K}{N}}}\left(\alpha_0\beta_0E_{1,\frac{\#J}{N}}(\lambda s)+\alpha_{-\#K}\beta_{\#K}(\lambda s)^{\frac{\#K}{N}}E_{1,1}(\lambda s)\right)\\
&=\frac{e^{-\lambda s}}{(\lambda s)^{\frac{\#K}{N}}}E_{1,\frac{\#J}{N}}(\lambda s)-1.
\end{aligned}\tag{5.4}$$

We can rewrite $E_{1,\frac{\#J}{N}}(\lambda s)$ as an integral by using Lemma 7.4. We obtain that

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[\mathbb{P}\{T(t)\in\mathrm{d}s,\,X(t)\le 0\}/\mathrm{d}s\right]\mathrm{d}t\\
&\qquad=\frac{e^{-\lambda s}}{(\lambda s)^{\frac{\#K}{N}}}\frac{1}{(\lambda s)^{\frac{\#J}{N}-1}}\left(e^{\lambda s}+\frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi}\int_0^\infty\frac{t^{1-\frac{\#J}{N}}}{t+1}e^{-\lambda st}\,\mathrm{d}t\right)-1\\
&\qquad=\frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi}e^{-\lambda s}\int_0^\infty\frac{t^{\frac{\#K}{N}}}{t+1}e^{-\lambda st}\,\mathrm{d}t=\frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi s^{\frac{\#K}{N}}}e^{-\lambda s}\int_0^\infty\frac{t^{\frac{\#K}{N}}}{t+s}e^{-\lambda t}\,\mathrm{d}t\\
&\qquad=\frac{\sin\big(\frac{\#K}{N}\pi\big)}{\pi s^{\frac{\#K}{N}}}\int_s^\infty\frac{(t-s)^{\frac{\#K}{N}}}{t}e^{-\lambda t}\,\mathrm{d}t.
\end{aligned}$$

From this, we extract, for $0 < s < t$,

$$\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \le 0\}/\mathrm{d}s = \frac{\sin\left(\frac{\#K}{N}\pi\right)}{\pi t} \left(\frac{t-s}{s}\right)^{\frac{\#K}{N}}. \tag{5.5}$$

We retrieve Theorem 14 of [11]. By integrating (5.5) with respect to s, we get

$$\mathbb{P}\{X(t) \le 0\} = \frac{\sin\left(\frac{\#K}{N}\pi\right)}{\pi t} \int_0^t \left(\frac{t-s}{s}\right)^{\frac{\#K}{N}} \mathrm{d}s = \frac{\sin\left(\frac{\#K}{N}\pi\right)}{\pi} B\left(\frac{\#K}{N}+1, 1-\frac{\#K}{N}\right) = \frac{\Gamma\left(\frac{\#K}{N}+1\right)}{\Gamma\left(\frac{\#K}{N}\right)}$$

which simplifies to $\mathbb{P}\{X(t) \le 0\} = \#K/N$. We retrieve (11) of [11].

Remark 5.2. An alternative expression for formula (5.1) is for $x \le 0$

$$\begin{aligned}\int_0^\infty & e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t \\ &= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}} \sum_{\substack{j\in J\\ k\in K}} A_j B_k \theta_k\, E_{\frac{1}{N},\frac{\#J}{N}}\left(\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda s}\right) e^{-\theta_k \sqrt[N]{\lambda}\, x}.\end{aligned} \tag{5.6}$$

In effect, by (5.1),

$$\begin{aligned}\int_0^\infty & e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t \\ &= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}} \sum_{\ell=0}^\infty \sum_{m=0}^{\#K} \sum_{k\in K} \alpha_{-m} B_k \theta_k^{m+1} \frac{(\lambda s)^{\ell+\frac{m}{N}}}{\Gamma\left(\ell+\frac{m+\#J}{N}\right)} e^{-\theta_k \sqrt[N]{\lambda}\, x} \\ &= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}} \sum_{\ell=0}^\infty \sum_{m=0}^{N-1} \sum_{\substack{j\in J\\ k\in K}} A_j B_k \theta_k \left(\frac{\theta_k}{\theta_j}\right)^m \frac{(\lambda s)^{\ell+\frac{m}{N}}}{\Gamma\left(\ell+\frac{m+\#J}{N}\right)} e^{-\theta_k \sqrt[N]{\lambda}\, x}.\end{aligned}$$

In the last displayed equality, we have extended the sum with respect to m to the range $0 \le m \le N-1$ because, by (2.9), the α_{-m}, $\#K+1 \le m \le N-1$, vanish. Let us introduce the index $r = \ell N + m$. Since $\left(\frac{\theta_k}{\theta_j}\right)^m = \left(\frac{\theta_k}{\theta_j}\right)^r$, we have

$$\int_0^\infty e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}} \sum_{\substack{j\in J\\ k\in K}} A_j B_k \theta_k \sum_{r=0}^\infty \frac{\left(\frac{\theta_k}{\theta_j}\sqrt[N]{\lambda s}\right)^r}{\Gamma\left(\frac{r+\#J}{N}\right)} e^{-\theta_k \sqrt[N]{\lambda}\, x}$$

which coincide with (5.6).

Example 5.1. Case $N = 2$. Suppose $x \le 0$. The first expression (5.1) reads

$$\int_0^\infty e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = \frac{e^{-\lambda s}}{\sqrt{s}} \left(E_{1,\frac12}(\lambda s) - \sqrt{\lambda s}\, E_{1,1}(\lambda s)\right) e^{\sqrt{\lambda}\, x}$$

while the second expression (5.6) reads

$$\int_0^\infty e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = \frac{e^{-\lambda s}}{\sqrt{s}}\, E_{\frac12,\frac12}\left(-\sqrt{\lambda s}\right) e^{\sqrt{\lambda}\, x}.$$

From Lemma 7.5, we have

$$E_{\frac12,\frac12}\left(-\sqrt{\lambda s}\right) = E_{1,\frac12}(\lambda s) - \sqrt{\lambda s}\, e^{\lambda s} \tag{5.7}$$

which proves the coincidence of both formulas. Moreover, from Lemma 7.5, for $x \le 0$,

$$\int_0^\infty e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = \left(\frac{e^{-\lambda s}}{\sqrt{\pi s}} - \sqrt{\lambda}\,\mathrm{Erfc}(\sqrt{\lambda s})\right) e^{\sqrt{\lambda}\, x}.$$

We retrieve formula 1.4.6, page 129, of [6].

Example 5.2. Case $N = 3$. We have for $x \le 0$, when $\kappa_3 = -1$:

$$\int_0^{\infty} e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = e^{\sqrt[3]{\lambda}\,x} \left(\frac{e^{-\lambda s}}{\sqrt[3]{s}}\, E_{1,\frac{2}{3}}(\lambda s) - \sqrt[3]{\lambda}\right)$$

and when $\kappa_3 = 1$:

$$\begin{aligned}\int_0^{\infty} & e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t \\ &= \frac{e^{-\lambda s+\frac{\sqrt[3]{\lambda}}{2}x}}{\sqrt{3}\,\sqrt[3]{\lambda s^2}} \left[\sqrt{3}\,\cos\!\left(\frac{\sqrt{3}\,\sqrt[3]{\lambda}\,x}{2}\right)\!\left(\sqrt[3]{\lambda s}\, E_{1,\frac{2}{3}}(\lambda s) - (\lambda s)^{2/3} e^{\lambda s}\right)\right. \\ &\quad \left. + \sin\!\left(\frac{\sqrt{3}\,\sqrt[3]{\lambda}\,x}{2}\right)\!\left(\sqrt[3]{\lambda s}\, E_{1,\frac{2}{3}}(\lambda s) + (\lambda s)^{2/3} e^{\lambda s} - 2E_{1,\frac{1}{3}}(\lambda s)\right)\right].\end{aligned}$$

Example 5.3. Case $N = 4$. We have, for $x \ge 0$,

$$\begin{aligned}\int_0^{\infty} & e^{-\lambda t}\, [\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t \\ &= \frac{\sqrt{2}\, e^{-\lambda s+\frac{\sqrt[4]{\lambda}}{\sqrt{2}}x}}{\sqrt[4]{\lambda}\,\sqrt{s}} \left[\cos\!\left(\frac{\sqrt[4]{\lambda}\,x}{\sqrt{2}}\right)\!\left(\sqrt[4]{\lambda s}\, E_{1,\frac{3}{4}}(\lambda s) - \sqrt{\lambda s}\, e^{\lambda s}\right) + \sin\!\left(\frac{\sqrt[4]{\lambda}\,x}{\sqrt{2}}\right)\!\left(\sqrt[4]{\lambda s}\, E_{1,\frac{3}{4}}(\lambda s) - E_{1,\frac{1}{2}}(\lambda s)\right)\right].\end{aligned}$$

5.2 The case $x \ge 0$

Theorem 5.2. *The Laplace transform with respect to t of the density of the couple $(T(t), X(t))$ is given, when $x \ge 0$, by*

$$\begin{aligned}\int_0^{\infty} & e^{-\lambda t}[\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t \\ &= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}}} \sum_{\substack{j\in J\\ k\in K}} A_j B_k \theta_k \int_0^s \sigma^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_k}{\theta_j}\,\sqrt[N]{\lambda\sigma}\right) I_{j,\#J-1}(s-\sigma; x)\,\mathrm{d}\sigma \qquad (5.8)\end{aligned}$$

where the function $I_{j,\#J-1}$ is defined by (2.15).

Proof
Recall (4.1) in the case $x \ge 0$:

$$\begin{aligned}\int_0^{\infty} & e^{-\lambda t}\, \big[\mathbb{E}\big(e^{-\nu T(t)},\, X(t) \in \mathrm{d}x\big)/\mathrm{d}x\big]\,\mathrm{d}t \\ &= \frac{1}{\lambda^{\frac{\#K-1}{N}} (\lambda+\nu)^{\frac{\#J-1}{N}}} \sum_{j\in J} A_j \theta_j \left(\sum_{k\in K} \frac{B_k \theta_k}{\theta_k \sqrt[N]{\lambda} - \theta_j \sqrt[N]{\lambda+\nu}}\right) e^{-\theta_j \sqrt[N]{\lambda+\nu}\,x}.\end{aligned}$$

We have to invert the quantity $\frac{e^{-\theta_j \sqrt[N]{\lambda+\nu}\,x}}{(\lambda+\nu)^{\frac{\#J-1}{N}} \left(\sqrt[N]{\lambda+\nu} - \frac{\theta_k}{\theta_j}\sqrt[N]{\lambda}\right)}$ with respect to ν. Recalling (5.2) and (2.16),

$$\begin{aligned}\frac{1}{\sqrt[N]{\lambda+\nu} - \beta} &= \int_0^{\infty} e^{-\nu s} \left(s^{\frac{1}{N}-1} e^{-\lambda s} E_{\frac{1}{N},\frac{1}{N}}\big(\beta\,\sqrt[N]{s}\,\big)\right) \mathrm{d}s, \\ \frac{e^{-\theta_j \sqrt[N]{\lambda+\nu}\,x}}{(\lambda+\nu)^{\frac{\#J-1}{N}}} &= \int_0^{\infty} e^{-\nu s} \left(e^{-\lambda s} I_{j,\#J-1}(s;x)\right) \mathrm{d}s,\end{aligned}$$

we get by convolution

$$\begin{aligned}&\frac{e^{-\theta_j \sqrt[N]{\lambda+\nu}\,x}}{(\lambda+\nu)^{\frac{\#J-1}{N}} \left(\sqrt[N]{\lambda+\nu} - \frac{\theta_k}{\theta_j}\sqrt[N]{\lambda}\,\right)} \\ &\quad = \int_0^{\infty} e^{-\nu s} \left(\int_0^s \sigma^{\frac{1}{N}-1} e^{-\lambda\sigma} E_{\frac{1}{N},\frac{1}{N}}\!\left(\frac{\theta_k}{\theta_j}\,\sqrt[N]{\lambda\sigma}\right) \times e^{-\lambda(s-\sigma)} I_{j,\#J-1}(s-\sigma; x)\,\mathrm{d}\sigma\right) \mathrm{d}s\end{aligned}$$

$$= \int_0^\infty e^{-\nu s} \left(e^{-\lambda s} \int_0^s \sigma^{\frac{1}{N}-1} E_{\frac{1}{N},\frac{1}{N}} \left(\frac{\theta_k}{\theta_j} \sqrt[N]{\lambda\sigma} \right) I_{j,\#J-1}(s-\sigma;x)\,\mathrm{d}\sigma \right) \mathrm{d}s.$$

This immediately yields (5.8). ■

Remark 5.3. Noticing that

$$E_{\frac{1}{N},\frac{1}{N}} \left(\frac{\theta_k}{\theta_j} \sqrt[N]{\lambda\sigma} \right) = \sum_{r=0}^{\infty} \frac{\theta_k^r}{\theta_j^r} \frac{(\lambda\sigma)^{\frac{r}{N}}}{\Gamma\big(\frac{r+1}{N}\big)} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{N-1} \frac{\theta_k^m}{\theta_j^m} \frac{(\lambda\sigma)^{\ell+\frac{m}{N}}}{\Gamma\big(\ell+\frac{m+1}{N}\big)} = \sum_{m=0}^{N-1} \frac{\theta_k^m}{\theta_j^m} (\lambda\sigma)^{\frac{m}{N}} E_{1,\frac{m+1}{N}}(\lambda\sigma)$$

and reminding that, from (2.8), the β_m, $1 \le m \le \#K-1$, vanish, we can rewrite (5.8) in the following form. For $x \ge 0$,

$$\int_0^\infty e^{-\lambda t} \left[\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\right] \mathrm{d}t$$

$$= -e^{-\lambda s} \sum_{m=\#K-1}^{N-1} \left(\sum_{k\in K} B_k \theta_k^{m+1} \right) \lambda^{\frac{m-\#K+1}{N}} \int_0^s \sigma^{\frac{m+1}{N}-1} E_{1,\frac{m+1}{N}}(\lambda\sigma) \left(\sum_{j\in J} \frac{A_j}{\theta_j^m} I_{j,\#J-1}(s-\sigma;x) \right) \mathrm{d}\sigma$$

$$= -e^{-\lambda s} \sum_{m=\#K}^{N} \beta_m \,\lambda^{\frac{m-\#K}{N}} \int_0^s \sigma^{\frac{m}{N}-1} E_{1,\frac{m}{N}}(\lambda\sigma)\, \Phi_m(s-\sigma;x)\,\mathrm{d}\sigma \tag{5.9}$$

with $\Phi_m(\tau;x) = \sum_{j\in J} \frac{A_j}{\theta_j^{m-1}} I_{j,\#J-1}(\tau;x)$.

Remark 5.4. Let us integrate (5.9) with respect to x on $[0,\infty)$. We first compute

$$\int_0^\infty \Phi_m(\tau;x)\,\mathrm{d}x = \frac{Ni}{2\pi} \left(\sum_{j\in J} \frac{A_j}{\theta_j^m} \right) \Big(e^{-i\frac{\#J}{N}\pi} - e^{i\frac{\#J}{N}\pi} \Big) \int_0^\infty \xi^{\#K-1} e^{-\tau\xi^N}\,\mathrm{d}\xi$$

$$= \frac{\Gamma\big(\frac{\#K}{N}\big)\sin\big(\frac{\#J}{N}\pi\big)}{\pi\tau^{\frac{\#K}{N}}} \alpha_{-m} = \frac{\alpha_{-m}}{\Gamma\big(\frac{\#J}{N}\big)\,\tau^{\frac{\#K}{N}}}.$$

Then, with the aid of (2.8) and (2.10), we get

$$\int_0^\infty e^{-\lambda t} \left[\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \ge 0\}/\mathrm{d}s\right] \mathrm{d}t$$

$$= -\frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J}{N}\big)} \left(\alpha_{-\#K}\,\beta_{\#K} \int_0^s \frac{\sigma^{\frac{\#K}{N}-1}}{(s-\sigma)^{\frac{\#K}{N}}} E_{1,\frac{\#K}{N}}(\lambda\sigma)\,\mathrm{d}\sigma + \alpha_{-N}\,\beta_N\,\lambda^{\frac{\#J}{N}} \int_0^s \frac{E_{1,1}(\lambda\sigma)}{(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma \right)$$

$$= \frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J}{N}\big)} \left(\int_0^s \frac{E_{1,\frac{\#K}{N}}(\lambda\sigma)}{\sigma^{\frac{\#J}{N}}(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma - \lambda^{\frac{\#J}{N}} \int_0^s \frac{e^{\lambda\sigma}}{(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma \right)$$

$$= \frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J}{N}\big)} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\Gamma\big(\ell+\frac{\#K}{N}\big)} \int_0^s \frac{\sigma^{\ell-\frac{\#J}{N}}}{(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma - \frac{\lambda^{\frac{\#J}{N}}}{\Gamma\big(\frac{\#J}{N}\big)} \int_0^s \frac{e^{-\lambda(s-\sigma)}}{(s-\sigma)^{\frac{\#K}{N}}}\,\mathrm{d}\sigma$$

$$= e^{-\lambda s} \sum_{\ell=0}^{\infty} \frac{(\lambda s)^\ell}{\ell!} - \frac{1}{\Gamma\big(\frac{\#J}{N}\big)} \int_0^{\lambda s} \frac{e^{-\sigma}}{\sigma^{\frac{\#K}{N}}}\,\mathrm{d}\sigma = 1 - \frac{1}{\Gamma\big(\frac{\#J}{N}\big)} \int_0^{\lambda s} \sigma^{\frac{\#J}{N}-1} e^{-\sigma}\,\mathrm{d}\sigma.$$

By Lemma 7.3, this simplifies into

$$\int_0^\infty e^{-\lambda t} \left[\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \ge 0\}/\mathrm{d}s\right] \mathrm{d}t = 1 - (\lambda s)^{\frac{\#J}{N}} e^{-\lambda s} E_{1,\frac{\#J}{N}+1}(\lambda s). \tag{5.10}$$

Now, using Lemma 7.4, we derive another representation for the foregoing Laplace transform:

$$\int_0^\infty e^{-\lambda t} \left[\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \ge 0\}/\mathrm{d}s\right] \mathrm{d}t = 1 - e^{-\lambda s} \left(e^{\lambda s} - \frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi} \int_0^\infty \frac{t^{-\frac{\#J}{N}}}{t+1} e^{-\lambda s t}\,\mathrm{d}t \right)$$

$$= \frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi} s^{\frac{\#J}{N}} \int_0^\infty \frac{e^{-\lambda(t+s)}}{(t+s)\,t^{\frac{\#J}{N}}}\,\mathrm{d}t$$

$$= \frac{\sin\big(\frac{\#J}{N}\pi\big)}{\pi} s^{\frac{\#J}{N}} \int_s^\infty \frac{e^{-\lambda t}}{t(t-s)^{\frac{\#J}{N}}}\,\mathrm{d}t.$$

As a result, we derive

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\geq 0\}/\mathrm{d}s = \frac{\sin\!\big(\frac{\#J}{N}\pi\big)}{\pi t}\left(\frac{s}{t-s}\right)^{\frac{\#J}{N}}, \qquad 0<s<t. \tag{5.11}$$

This is formula (11) of [11].

Remark 5.5. If we add (5.4) and (5.10), we find

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s\}/\mathrm{d}s]\,\mathrm{d}t = e^{-\lambda s}\left(\sum_{\ell=0}^{\infty}\frac{(\lambda s)^{\ell+\frac{\#J}{N}-1}}{\Gamma\big(\ell+\frac{\#J}{N}\big)} - \sum_{\ell=0}^{\infty}\frac{(\lambda s)^{\ell+\frac{\#J}{N}}}{\Gamma\big(\ell+\frac{\#J}{N}+1\big)}\right) = \frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J}{N}\big)(\lambda s)^{\frac{\#K}{N}}}.$$

It is easy to invert this Laplace transform. Indeed,

$$\begin{aligned}\frac{e^{-\lambda s}}{\lambda^{\frac{\#K}{N}}} = \frac{e^{-\lambda s}}{\Gamma\big(\frac{\#K}{N}\big)}\int_0^\infty t^{\frac{\#K}{N}-1}e^{-\lambda t}\,\mathrm{d}t &= \frac{1}{\Gamma\big(\frac{\#K}{N}\big)}\int_s^\infty (t-s)^{\frac{\#K}{N}-1}e^{-\lambda t}\,\mathrm{d}t\\ &= \frac{1}{\pi}\,\Gamma\!\left(\frac{\#J}{N}\right)\,\sin\!\left(\frac{\#J}{N}\pi\right)\int_s^\infty \frac{e^{-\lambda t}}{(t-s)^{\frac{\#J}{N}}}\,\mathrm{d}t.\end{aligned}$$

This implies that

$$\mathbb{P}\{T(t)\in \mathrm{d}s\}/\mathrm{d}s = \frac{\sin\!\big(\frac{\#J}{N}\pi\big)}{\pi}\,\frac{\mathbb{1}_{(0,t)}(s)}{s^{\frac{\#K}{N}}(t-s)^{\frac{\#J}{N}}}$$

which can be also obtained by adding directly (5.5) and (5.11). Thus, we retrieve the famous counterpart to the Paul Lévy's arc-sine law stated in [11] (Corollary 9).

Remark 5.6. For $x=0$, using formula (5.1) which is valid for $x\leq 0$, we get, by (2.8), (2.9) and (2.10),

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\,\mathbb{P}\{T(t)\in \mathrm{d}s,\, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\Big|_{x=0}\,\mathrm{d}t\\ &\quad= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}}s^{\frac{\#K}{N}}}\sum_{m=0}^{\#K}\alpha_{-m}\,\beta_{m+1}\,(\lambda s)^{\frac{m}{N}}E_{1,\frac{m+\#J}{N}}(\lambda s)\\ &\quad= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}}s^{\frac{\#K}{N}}}\Big(\alpha_{1-\#K}\,\beta_{\#K}\,(\lambda s)^{\frac{\#K-1}{N}}E_{1,1-\frac{1}{N}}(\lambda s)+\alpha_{-\#K}\,\beta_{\#K+1}\,(\lambda s)^{\frac{\#K}{N}}E_{1,1}(\lambda s)\Big)\\ &\quad= \frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}}s^{\frac{\#K}{N}}}\left(\sum_{j\in J}\theta_j(\lambda s)^{\frac{\#K-1}{N}}E_{1,1-\frac{1}{N}}(\lambda s)+\sum_{k\in K}\theta_k(\lambda s)^{\frac{\#K}{N}}e^{\lambda s}\right)\\ &\quad= \left(\sum_{j\in J}\theta_j\right)\frac{e^{-\lambda s}}{\sqrt[N]{s}}\Big(E_{1,1-\frac{1}{N}}(\lambda s)-\sqrt[N]{\lambda s}\,e^{\lambda s}\Big).\end{aligned} \tag{5.12}$$

On the other hand, with formula (5.9) which is valid for $x\geq 0$,

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\,\mathbb{P}\{T(t)\in \mathrm{d}s,\, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\Big|_{x=0}\,\mathrm{d}t\\ &\qquad= -e^{-\lambda s}\sum_{m=\#K}^{N}\beta_m\,\lambda^{\frac{m-\#K}{N}}\int_0^s \sigma^{\frac{m}{N}-1}E_{1,\frac{m}{N}}(\lambda\sigma)\,\Phi_m(s-\sigma;0)\,\mathrm{d}\sigma\end{aligned} \tag{5.13}$$

with

$$\begin{aligned}\Phi_m(\tau;0) &= \frac{Ni}{2\pi}\left(\sum_{j\in J}\frac{A_j}{\theta_j^{m-1}}\right)\Big(e^{-i\frac{\#J-1}{N}\pi}-e^{i\frac{\#J-1}{N}\pi}\Big)\int_0^\infty \xi^{\#K}e^{-\tau\xi^N}\,\mathrm{d}\xi\\ &= \frac{\Gamma\big(\frac{\#K+1}{N}\big)\sin\!\big(\frac{\#J-1}{N}\pi\big)}{\pi\,\tau^{\frac{\#K+1}{N}}}\,\alpha_{1-m} = \frac{\alpha_{1-m}}{\Gamma\big(\frac{\#J-1}{N}\big)\,\tau^{\frac{\#K+1}{N}}}.\end{aligned}$$

In view of (2.8), (2.9) and (2.10), we have

$$\int_0^\infty e^{-\lambda t}\,\mathbb{P}\{T(t)\in \mathrm{d}s,\,X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\Big|_{x=0}\mathrm{d}t$$

$$\begin{aligned}
&= \frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J-1}{N}\big)}\left[\left(\sum_{j\in J}\theta_j\right)\int_0^s \frac{\sigma^{\frac{\#K}{N}-1}}{(s-\sigma)^{\frac{\#K+1}{N}}}\,E_{1,\frac{\#K}{N}}(\lambda\sigma)\,\mathrm{d}\sigma\right.\\
&\quad\left.+\left(\sum_{k\in K}\theta_k\right)\sqrt[N]{\lambda}\int_0^s \frac{\sigma^{\frac{\#K+1}{N}-1}}{(s-\sigma)^{\frac{\#K+1}{N}}}\,E_{1,\frac{\#K+1}{N}}(\lambda\sigma)\,\mathrm{d}\sigma\right]\\
&= \left(\sum_{j\in J}\theta_j\right)\frac{e^{-\lambda s}}{\Gamma\big(\frac{\#J-1}{N}\big)}\,e^{-\lambda s}\left(\sum_{\ell=0}^{\infty}\frac{B\big(\ell+\frac{\#K}{N},1-\frac{\#K+1}{N}\big)}{\Gamma\big(\ell+\frac{\#K}{N}\big)}\,\lambda^\ell s^{\ell-\frac{1}{N}}\right.\\
&\quad\left.-\sqrt[N]{\lambda}\sum_{\ell=0}^{\infty}\frac{B\big(\ell+\frac{\#K+1}{N},1-\frac{\#K+1}{N}\big)}{\Gamma\big(\ell+\frac{\#K+1}{N}\big)}\,(\lambda s)^\ell\right)\\
&= \left(\sum_{j\in J}\theta_j\right)\frac{e^{-\lambda s}}{\sqrt[N]{s}}\left(E_{1,1-\frac{1}{N}}(\lambda s)-\sqrt[N]{\lambda s}\,e^{\lambda s}\right).
\end{aligned}$$

Thus, we have checked that the two different formulas (5.12) and (5.13) lead to the same result.

Example 5.4. Case $N=2$. Suppose $x\geq 0$. Formula (5.8) reads, with the numerical values of Example 2.1,

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = e^{-\lambda s}\int_0^s \frac{1}{\sqrt{\sigma}}\,E_{\frac12,\frac12}\big(-\sqrt{\lambda\sigma}\big)\,I_{1,0}(s-\sigma;x)\,\mathrm{d}\sigma$$

while formula (5.9) gives, because of $\Phi_1=\Phi_2=I_{1,0}$ and (5.7),

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s,\,X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$

$$\begin{aligned}
&= e^{-\lambda s}\left(\int_0^s \frac{1}{\sqrt{\sigma}}\,E_{1,\frac12}(\lambda\sigma)\,\Phi_1(s-\sigma;x)\,\mathrm{d}\sigma-\sqrt{\lambda}\int_0^s E_{1,1}(\lambda\sigma)\,\Phi_2(s-\sigma;x)\,\mathrm{d}\sigma\right)\\
&= e^{-\lambda s}\int_0^s \frac{1}{\sqrt{\sigma}}\left(E_{1,\frac12}(\lambda\sigma)-\sqrt{\lambda\sigma}\,e^{\lambda\sigma}\right)I_{1,0}(s-\sigma;x)\,\mathrm{d}\sigma\\
&= e^{-\lambda s}\int_0^s \frac{1}{\sqrt{\sigma}}\,E_{\frac12,\frac12}\big(-\sqrt{\lambda\sigma}\big)\,I_{1,0}(s-\sigma;x)\,\mathrm{d}\sigma.
\end{aligned}$$

We have checked that the two different representations (5.8) and (5.9) lead to the same result. Let us pursue the computations. In view of (2.17) and Lemma 7.5, we get

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s,\,X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$

$$\begin{aligned}
&= \frac{xe^{-\lambda s}}{2\sqrt{\pi}}\int_0^s \frac{e^{-\frac{x^2}{4(s-\sigma)}}}{\sqrt{\sigma}\,(s-\sigma)^{3/2}}\,E_{\frac12,\frac12}\big(-\sqrt{\lambda\sigma}\big)\,\mathrm{d}\sigma\\
&= \frac{xe^{-\lambda s}}{2\sqrt{\pi}}\int_0^s \frac{e^{-\frac{x^2}{4\sigma}}}{\sigma^{3/2}\sqrt{s-\sigma}}\,E_{\frac12,\frac12}\big(-\sqrt{\lambda(s-\sigma)}\big)\,\mathrm{d}\sigma\\
&= \frac{xe^{-\lambda s}}{2\sqrt{\pi}}\int_0^s \frac{e^{-\frac{x^2}{4\sigma}}}{\sigma^{3/2}\sqrt{s-\sigma}}\left(\frac{1}{\sqrt{\pi}}-\sqrt{\lambda(s-\sigma)}\,e^{\lambda(s-\sigma)}\,\mathrm{Erfc}\big(\sqrt{\lambda(s-\sigma)}\big)\right)\mathrm{d}\sigma\\
&= \frac{xe^{-\lambda s}}{2\sqrt{\pi}}\left[\frac{1}{\sqrt{\pi}}\int_0^s \frac{e^{-\frac{x^2}{4\sigma}}}{\sigma^{3/2}\sqrt{s-\sigma}}\,\mathrm{d}\sigma-\frac{2\sqrt{\lambda}}{\sqrt{\pi}}\int_0^s \frac{e^{\lambda(s-\sigma)-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_{\sqrt{\lambda(s-\sigma)}}^\infty e^{-\xi^2}\,\mathrm{d}\xi\right)\mathrm{d}\sigma\right].
\end{aligned}$$

The first integral in the last displayed equality writes, with the change of variable $\sigma=s^2/\tau$,

$$\int_0^s \frac{e^{-\frac{x^2}{4\sigma}}}{\sigma^{3/2}\sqrt{s-\sigma}}\,\mathrm{d}\sigma=\frac{1}{s^{3/2}}\int_s^\infty \frac{e^{-\frac{x^2}{4s^2}\tau}}{\sqrt{\tau-s}}\,\mathrm{d}\tau=\frac{e^{-\frac{x^2}{4s}}}{s^{3/2}}\int_0^\infty \frac{e^{-\frac{x^2}{4s^2}\tau}}{\sqrt{\tau}}\,\mathrm{d}\tau=\frac{2\sqrt{\pi}}{\sqrt{s}\,x}\,e^{-\frac{x^2}{4s}},$$

and then

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$
$$= \frac{1}{\sqrt{\pi s}}\,e^{-\lambda s-\frac{x^2}{4s}} - \frac{x\sqrt{\lambda}}{\pi}\int_0^s \frac{e^{-\lambda\sigma-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_{\sqrt{\lambda(s-\sigma)}}^\infty e^{-\xi^2}\,\mathrm{d}\xi\right)\mathrm{d}\sigma.$$

The computation of the integral lying on the right-hand side of the foregoing equality being cumbersome is postponed to Lemma 7.7 in the appendix. The final result is, for $x\ge 0$,

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t = \frac{e^{-\lambda s-\frac{x^2}{4s}}}{\sqrt{\pi s}} - \sqrt{\lambda}\,e^{\sqrt{\lambda}\,x}\,\mathrm{Erfc}\!\left(\frac{x}{2\sqrt{s}}+\sqrt{\lambda s}\right).$$

This is formula 1.4.6, page 129, of [6].

Example 5.5. Case $N=3$. For $x\ge 0$, (5.8) supplies with the numerical values of Example 2.2, when $\kappa_3=-1$,

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s,\ X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$
$$= \frac{e^{-\lambda s}}{\sqrt{3}}\left(e^{\frac{i\pi}{6}}\int_0^s \sigma^{-2/3}E_{\frac13,\frac13}\big(-e^{-i\frac{\pi}{3}}\sqrt[3]{\lambda\sigma}\big)\,I_{1,1}(s-\sigma;x)\,\mathrm{d}\sigma + e^{-\frac{i\pi}{6}}\int_0^s \sigma^{-2/3}E_{\frac13,\frac13}\big(-e^{i\frac{\pi}{3}}\sqrt[3]{\lambda\sigma}\big)\,I_{2,1}(s-\sigma;x)\,\mathrm{d}\sigma\right)$$

and when $\kappa_3=1$,

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s,\ X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$
$$= \frac{i\,e^{-\lambda s}}{\sqrt{3}\,\sqrt[3]{\lambda}}\left(\int_0^s \sigma^{-2/3}E_{\frac13,\frac13}\big(e^{-i\frac{2\pi}{3}}\sqrt[3]{\lambda\sigma}\big)\,I_{1,0}(s-\sigma;x)\,\mathrm{d}\sigma - \int_0^s \sigma^{-2/3}E_{\frac13,\frac13}\big(e^{i\frac{2\pi}{3}}\sqrt[3]{\lambda\sigma}\big)\,I_{1,0}(s-\sigma;x)\,\mathrm{d}\sigma\right).$$

The functions $I_{1,0}$, $I_{1,1}$ and $I_{2,1}$ above are respectively given by (2.18), (2.19) and (2.20).

Example 5.6. Case $N=4$. For $x\ge 0$, (5.8) supplies, with the numerical values of Example 2.3,

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t)\in \mathrm{d}s,\ X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\,\mathrm{d}t$$
$$\begin{aligned}= -\frac{e^{-\lambda s}}{2\sqrt[4]{\lambda}}\bigg(& e^{i\frac{\pi}{4}}\int_0^s \sigma^{-3/4}E_{\frac14,\frac14}\big(-\sqrt[4]{\lambda\sigma}\big)\,I_{1,1}(s-\sigma;x)\,\mathrm{d}\sigma\\ &+e^{-i\frac{3\pi}{4}}\int_0^s \sigma^{-3/4}E_{\frac14,\frac14}\big(-i\sqrt[4]{\lambda\sigma}\big)\,I_{1,1}(s-\sigma;x)\,\mathrm{d}\sigma\\ &+e^{i\frac{3\pi}{4}}\int_0^s \sigma^{-3/4}E_{\frac14,\frac14}\big(i\sqrt[4]{\lambda\sigma}\big)\,I_{2,1}(s-\sigma;x)\,\mathrm{d}\sigma\\ &+e^{-i\frac{\pi}{4}}\int_0^s \sigma^{-3/4}E_{\frac14,\frac14}\big(-\sqrt[4]{\lambda\sigma}\big)\,I_{2,1}(s-\sigma;x)\,\mathrm{d}\sigma\bigg).\end{aligned}$$

The functions $I_{1,1}$ and $I_{2,1}$ above are given by (2.21).

6 Inverting with respect to λ

In this section, we perform the last inversion in $F(\lambda,\mu,\nu)$ in order to derive the distribution of the couple $(T(t),X(t))$. As in the previous section, we treat separately the two cases $x\le 0$ and $x\ge 0$.

6.1 The case $x \le 0$

Theorem 6.1. *The distribution of the couple $(T(t), X(t))$ is given, for $x \le 0$, by*

$$\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/\mathrm{d}s\,\mathrm{d}x$$

$$= -\frac{Ni}{2\pi} \sum_{m=0}^{\#K} \alpha_{-m} s^{\frac{m-\#K}{N}} \int_0^\infty \xi^{m+\#J} e^{-(t-s)\xi^N} \mathcal{K}_m(x\xi)\, E_{1,\frac{m+\#J}{N}}(-s\xi^N)\,\mathrm{d}\xi \tag{6.1}$$

where

$$\mathcal{K}_m(z) = e^{-i\frac{\#K-m-1}{N}\pi} \sum_{k\in K} B_k\theta_k^{m+1} e^{-\theta_k e^{i\frac{\pi}{N}} z} - e^{i\frac{\#K-m-1}{N}\pi} \sum_{k\in K} B_k\theta_k^{m+1} e^{-\theta_k e^{-i\frac{\pi}{N}} z}.$$

Proof

Assume $x \le 0$. Recalling (5.1), we have

$$\int_0^\infty e^{-\lambda t}\,[\mathbb{P}\{T(t) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)]\mathrm{d}t$$

$$\begin{aligned}
&= -\frac{e^{-\lambda s}}{\lambda^{\frac{\#K-1}{N}} s^{\frac{\#K}{N}}} \sum_{m=0}^{\#K} \alpha_{-m} (\lambda s)^{\frac{m}{N}} E_{1,\frac{m+\#J}{N}}(\lambda s) \sum_{k\in K} B_k\theta_k^{m+1} e^{-\theta_k \sqrt[N]{\lambda}\, x} \\
&= -\sqrt[N]{\lambda}\, e^{-\lambda s} \sum_{m=0}^{\#K} \alpha_{-m} \sum_{\ell=0}^{\infty} \frac{(\lambda s)^{\ell+\frac{m-\#K}{N}}}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)} \sum_{k\in K} B_k\theta_k^{m+1} e^{-\theta_k \sqrt[N]{\lambda}\, x} \\
&= -\sum_{\ell=0}^{\infty}\sum_{m=0}^{\#K} \alpha_{-m} \frac{s^{\ell+\frac{m-\#K}{N}}}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)} \sum_{k\in K} B_k\theta_k^{m+1}\, \lambda^{\ell+\frac{m-\#K+1}{N}} e^{-\lambda s-\theta_k \sqrt[N]{\lambda}\, x}.
\end{aligned} \tag{6.2}$$

We need to invert the quantity $\lambda^{\ell+\frac{m-\#K+1}{N}} e^{-\lambda s-\theta_k \sqrt[N]{\lambda}\, x}$ for $\ell \ge 0$ and $0 \le m \le \#K$ with respect to λ. We intend to use (2.16) which is valid for $0 \le m \le N-1$. Actually (2.16) holds true also for $m \le 0$; the proof of this claim is postponed to Lemma 7.8 in the appendix. As a byproduct, for any $\ell \ge 0$ and $0 \le m \le \#K$,

$$\begin{aligned}
\lambda^{\ell+\frac{m-\#K+1}{N}} e^{-\lambda s-\theta_k \sqrt[N]{\lambda}\, x} &= e^{-\lambda s} \int_0^\infty e^{-\lambda u} I_{k,\#K-\ell N-m-1}(u;x)\,\mathrm{d}u \\
&= \int_s^\infty e^{-\lambda t} I_{k,\#K-\ell N-m-1}(t-s;x)\,\mathrm{d}t.
\end{aligned} \tag{6.3}$$

Then, by putting (6.3) into (6.2) and next by eliminating the Laplace transform with respect to λ, we extract

$$\mathbb{P}\{T(s) \in \mathrm{d}s,\, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$$

$$\begin{aligned}
&= -\sum_{\ell=0}^{\infty}\sum_{m=0}^{\#K} \alpha_{-m} \frac{s^{\ell+\frac{m-\#K}{N}}}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)} \sum_{k\in K} B_k\theta_k^{m+1}\, I_{k,\#K-\ell N-m-1}(t-s;x) \\
&= -\frac{Ni}{2\pi}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\#K} \alpha_{-m} \frac{s^{\ell+\frac{m-\#K}{N}}}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)} \\
&\quad\times \sum_{k\in K} B_k\theta_k^{m+1} \bigg(e^{-i\frac{\#K-\ell N-m-1}{N}\pi} \int_0^\infty \xi^{N-\#K+\ell N+m}\, e^{-(t-s)\xi^N-\theta_k e^{i\frac{\pi}{N}} x\xi}\,\mathrm{d}\xi \\
&\quad - e^{i\frac{\#K-\ell N-m-1}{N}\pi} \int_0^\infty \xi^{N-\#K+\ell N+m}\, e^{-(t-s)\xi^N-\theta_k e^{-i\frac{\pi}{N}} x\xi}\,\mathrm{d}\xi \bigg) \\
&= -\frac{Ni}{2\pi}\sum_{m=0}^{\#K} \alpha_{-m} s^{\frac{m-\#K}{N}} \sum_{k\in K} B_k\theta_k^{m+1} \\
&\quad\times \bigg(e^{-i\frac{\#K-m-1}{N}\pi} \int_0^\infty \bigg(\sum_{\ell=0}^{\infty} \frac{\big(-s\xi^N\big)^\ell}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)}\bigg) \xi^{m+\#J}\, e^{-(t-s)\xi^N-\theta_k e^{i\frac{\pi}{N}} x\xi}\,\mathrm{d}\xi \\
&\quad - e^{i\frac{\#K-m-1}{N}\pi} \int_0^\infty \bigg(\sum_{\ell=0}^{\infty} \frac{\big(-s\xi^N\big)^\ell}{\Gamma\big(\ell+\frac{m+\#J}{N}\big)}\bigg) \xi^{m+\#J}\, e^{-(t-s)\xi^N-\theta_k e^{-i\frac{\pi}{N}} x\xi}\,\mathrm{d}\xi \bigg)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{Ni}{2\pi} \sum_{m=0}^{\#K} \alpha_{-m} s^{\frac{m-\#K}{N}} \sum_{k\in K} B_k \theta_k^{m+1} \\
&\quad \times \left(e^{-i\frac{\#K-m-1}{N}\pi} \int_0^{\infty} \xi^{m+\#J} \, e^{-(t-s)\,\xi^N - \theta_k e^{i\frac{\pi}{N}} x\xi} \, E_{1,\frac{m+\#J}{N}}\big(-s\xi^N\big) \,\mathrm{d}\xi \right. \\
&\quad \left. - e^{i\frac{\#K-m-1}{N}\pi} \int_0^{\infty} \xi^{m+\#J} \, e^{-(t-s)\,\xi^N - \theta_k e^{-i\frac{\pi}{N}} x\xi} \, E_{1,\frac{m+\#J}{N}}\big(-s\xi^N\big) \,\mathrm{d}\xi \right).
\end{aligned}
$$

The proof of (6.1) is established. ■

Remark 6.1. Let us integrate (6.1) with respect to x on $(-\infty, 0]$. We first compute, by using (2.8),

$$
\begin{aligned}
\int_{-\infty}^{0} \mathcal{K}_m(x\xi) \,\mathrm{d}x &= -\frac{1}{\xi} \left(\sum_{k\in K} B_k \theta_k^m \right) \Big(e^{-i\frac{\#K-m}{N}\pi} - e^{i\frac{\#K-m}{N}\pi} \Big) \\
&= \frac{2i}{\xi} \sin\!\Big(\frac{\#K-m}{N}\pi \Big) \beta_m = \begin{cases} 0 & \text{if } 1 \le m \le \#K, \\ \dfrac{2i}{\xi} \sin\!\Big(\dfrac{\#K}{N}\pi \Big) & \text{if } m = 0. \end{cases}
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\le 0\}/\mathrm{d}s &= \frac{N \sin\big(\frac{\#K}{N}\pi\big)}{\pi s^{\frac{\#K}{N}}} \int_0^{\infty} \xi^{\#J-1} \, e^{-(t-s)\xi^N} \, E_{1,\frac{\#J}{N}}\big(-s\xi^N\big) \,\mathrm{d}\xi \\
&= \frac{N \sin\big(\frac{\#K}{N}\pi\big)}{\pi s^{\frac{\#K}{N}}} \sum_{\ell=0}^{\infty} \frac{(-s)^\ell}{\Gamma\big(\ell + \frac{\#J}{N}\big)} \int_0^{\infty} \xi^{\ell N + \#J - 1} e^{-(t-s)\xi^N} \,\mathrm{d}\xi \\
&= \frac{\sin\big(\frac{\#K}{N}\pi\big)}{\pi s^{\frac{\#K}{N}} (t-s)^{\frac{\#J}{N}}} \sum_{\ell=0}^{\infty} \Big(-\frac{s}{t-s} \Big)^{\ell} = \frac{\sin\big(\frac{\#K}{N}\pi\big)}{\pi t} \Big(\frac{t-s}{s} \Big)^{\frac{\#K}{N}}.
\end{aligned}
$$

We retrieve (5.5).

Remark 6.2. Let us evaluate $\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$ at $x=0$. For $0 \le m \le \#K$,

$$
\mathcal{K}_m(0) = e^{-i\frac{\#K-m-1}{N}\pi} \sum_{k\in K} B_k \theta_k^{m+1} - e^{i\frac{\#K-m-1}{N}\pi} \sum_{k\in K} B_k \theta_k^{m+1} = -2i \sin\!\Big(\frac{\#K-m-1}{N}\,\pi \Big) \beta_{m+1}.
$$

Observing that $\sin\big(\frac{\#K-m-1}{N}\pi\big) = 0$ if $m = \#K-1$, in view of (2.8), (2.9) and (2.10), we get

$$
\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s,\, X(t)\in \mathrm{d}x\}/\mathrm{d}s\Big|_{x=0} \\
&\qquad = \frac{N}{\pi} \sin\!\Big(\frac{\pi}{N}\Big) \alpha_{-\#K} \beta_{\#K+1} \int_0^{\infty} \xi^N \, e^{-(t-s)\xi^N} E_{1,1}\big(-s\xi^N\big) \,\mathrm{d}\xi \\
&\qquad = \frac{N}{\pi} \sin\!\Big(\frac{\pi}{N}\Big) \Bigg(\sum_{j\in J} \theta_j \Bigg) \int_0^{\infty} \xi^N \, e^{-t\xi^N} \mathrm{d}\xi = \frac{\sin\big(\frac{\pi}{N}\big)\Gamma\big(\frac{1}{N}\big)}{N\pi\, t^{1+\frac{1}{N}}} \sum_{j\in J} \theta_j.
\end{aligned}
$$

Thanks to (2.4) and (2.12), we see that

$$
\mathbb{P}\{T(t)\in \mathrm{d}s,\, X(t)\in \mathrm{d}x\}/\mathrm{d}s\Big|_{x=0} = \frac{1}{t}\, p(t;0)
$$

and we deduce

$$
\mathbb{P}\{T(t)\in \mathrm{d}s | X(t)=0\}/\mathrm{d}s = \frac{\mathbb{1}_{(0,t)}(s)}{t},
$$

that is, $(T(t)|X(t)=0)$ has the uniform law on $(0,t)$. This is Theorem 2.13 of [11].

6.2 The case $x \geq 0$

The case $x \geq 0$ can be related to the case $x \leq 0$ by using the duality. Let us introduce the dual process $(X_t^*)_{t\geq 0}$ of $(X_t)_{t\geq 0}$ defined as $X_t^* = -X_t$ for any $t \geq 0$. It is known that (see [11]):

- If N is even, the processes X and X^* are identical in distribution (because of the symmetry of the heat kernel p): $X^* \stackrel{d}{=} X$;
- If N is odd, we have the equalities in distribution $(X^+)^* \stackrel{d}{=} X^-$ and $(X^-)^* \stackrel{d}{=} X^+$ where X^+ is the pseudo-process associated with $\kappa_N = +1$ and X^- the one associated with $\kappa_N = -1$.

When N is even, we have $\{-\theta_j, j \in J\} = \{\theta_k, k \in K\}$. In this case, for any $j \in J$, there exists a unique $k \in K$ such that $\theta_j = -\theta_k$ and then

$$A_j = \prod_{i \in J\setminus\{j\}} \frac{\theta_i}{\theta_i - \theta_j} = \prod_{i \in K\setminus\{k\}} \frac{-\theta_i}{-\theta_i + \theta_k} = \prod_{i \in K\setminus\{k\}} \frac{\theta_i}{\theta_i - \theta_k} = B_k$$

and

$$\alpha_m = \sum_{j\in J} A_j \theta_j^m = \sum_{k \in K} B_k(-\theta_k)^m = (-1)^m \beta_m.$$

When N is odd, we distinguish the roots of κ_N in the cases $\kappa_N = +1$ and $\kappa_N = -1$:

- For $\kappa_N = +1$, let θ_i^+, $1 \leq i \leq N$, denote the roots of 1 and set $J^+ = \{i \in \{1,\dots,N\} : \Re(\theta_i^+) > 0\}$ and $K^+ = \{i \in \{1,\dots,N\} : \Re(\theta_i^+) < 0\}$;
- For $\kappa_N = -1$, let θ_i^-, $1 \leq i \leq N$, denote the roots of -1 and set $J^- = \{i \in \{1,\dots,N\} : \Re(\theta_i^-) > 0\}$ and $K^- = \{i \in \{1,\dots,N\} : \Re(\theta_i^-) < 0\}$.

We have $\{\theta_j^-, i \in J^-\} = \{-\theta_k^+, k \in K^+\}$ and $\{\theta_k^-, k \in K^-\} = \{-\theta_j^+, j \in J^+\}$. In this case, for any $j \in J^-$, there exists a unique $k \in K^+$ such that $\theta_j^- = -\theta_k^+$ and then

$$A_j^- = \prod_{i \in J^-\setminus\{j\}} \frac{\theta_i^-}{\theta_i^- - \theta_j^-} = \prod_{i \in K^+\setminus\{k\}} \frac{-\theta_i^+}{-\theta_i^+ - \theta_k^+} = \prod_{i \in K^+\setminus\{k\}} \frac{\theta_i^+}{\theta_i^+ - \theta_k^+} = B_k^+$$

and similarly $A_j^+ = B_k^-$. Moreover, we have

$$\alpha_m^- = \sum_{j \in J^-} A_j^-(\theta_j^-)^m = \sum_{k\in K^+} B_k^+(-\theta_k^+)^m = (-1)^m \sum_{k \in K^+} B_k^+(\theta_k^+)^m = (-1)^m \beta_m^+$$

and similarly $\alpha_m^+ = (-1)^m \beta_m^-$.

Now, concerning the connection between sojourn time and duality, we have the following fact. Set

$$\tilde{T}(t) = \int_0^t 1\!\!1_{(0,+\infty)}(X(u))\,\mathrm{d}u \quad \text{and} \quad T^*(t) = \int_0^t 1\!\!1_{[0,+\infty)}(X^*(u))\,\mathrm{d}u.$$

Since Spitzer's identity holds true interchanging the closed interval $[0,+\infty)$ and the open interval $(0,+\infty)$, it is easy to see that $T(t)$ and $\tilde{T}(t)$ have the same distribution. On the other hand, we have

$$\tilde{T}(t) = \int_0^t 1\!\!1_{(0,+\infty)}(X(u))\,\mathrm{d}u = \int_0^t 1\!\!1_{(-\infty,0)}(X^*(u))\,\mathrm{d}u = \int_0^t [1 - 1\!\!1_{[0,+\infty)}(X^*(u))]\,\mathrm{d}u = t - T^*(t).$$

We then deduce that $T(t)$ and $t - T^*(t)$ have the same distribution. Consequently, we can state the lemma below.

Lemma 6.1. *The following identity holds:*

$$\mathbb{P}\{T(t) \in \mathrm{d}s, X(t) \in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x) = \mathbb{P}\{T^*(t) \in \mathrm{d}(t-s), X^*(t) \in \mathrm{d}(-x)\}/(\mathrm{d}s\,\mathrm{d}x).$$

As a result, the following result ensues.

Theorem 6.2. *Assume N is even. The distribution of $(T(t),X(t))$ is given, for $x\ge 0$, by*

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=\frac{Ni}{2\pi}\sum_{m=0}^{\#J}\beta_{-m}\,(t-s)^{\frac{m-\#J}{N}}\int_0^\infty \xi^{m+\#K}\,e^{-s\xi^N}\,\mathcal{J}_m(x\xi)\,E_{1,\frac{m+\#K}{N}}\big(-(t-s)\xi^N\big)\,\mathrm{d}\xi \tag{6.4}$$

where

$$\mathcal{J}_m(z)=e^{-i\frac{\#J-m-1}{N}\pi}\sum_{j\in J}A_j\theta_j^{m+1}e^{-\theta_j e^{i\frac{\pi}{N}}z}-e^{i\frac{\#J-m-1}{N}\pi}\sum_{j\in J}A_j\theta_j^{m+1}e^{-\theta_j e^{-i\frac{\pi}{N}}z}.$$

Proof

When N is even, we know that X^* is identical in distribution to X and $(T^*(t),X^*(t))$ is then distributed like $(T(t),X(t))$. Thus, by (6.1) and Lemma 6.1, for $x\ge 0$,

$$\mathbb{P}\{T(t)\in \mathrm{d}s,\ X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=\mathbb{P}\{T(t)\in \mathrm{d}(t-s),\ X(t)\in \mathrm{d}(-x)\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=-\frac{Ni}{2\pi}\sum_{m=0}^{\#K}\alpha_{-m}(t-s)^{\frac{m-\#K}{N}}\int_0^\infty \xi^{m+\#J}e^{-s\xi^N}\,\mathcal{K}_m(-x\xi)\,E_{1,\frac{m+\#J}{N}}(-(t-s)\xi^N)\,\mathrm{d}\xi.$$

The discussion preceding Lemma 6.1 shows that

$$\mathcal{K}_m(z)=e^{-i\frac{\#J-m-1}{N}\pi}\sum_{j\in J}A_j(-\theta_j)^{m+1}e^{\theta_j e^{i\frac{\pi}{N}}z}-e^{i\frac{\#J-m-1}{N}\pi}\sum_{j\in J}A_j(-\theta_j)^{m+1}e^{\theta_j e^{-i\frac{\pi}{N}}z}.$$

We see that $\mathcal{K}_m(z)=(-1)^{m+1}\mathcal{J}_m(-z)$ where the function $\mathcal{J}_m$ is written in Theorem 6.2. Finally, by replacing α_{-m} by $(-1)^m\beta_{-m}$ and $\#J$, $\#K$ by $\#K$, $\#J$ respectively (which actually coincide since N is even), (6.4) ensues. ■

If N is odd, although the results are not justified, similar formulas can be stated. We find it interesting to produce them here. We set $T^\pm(t)=\int_0^t \mathbb{1}_{[0,+\infty)}(X^\pm(u))\,\mathrm{d}u$.

Theorem 6.3. *Suppose that N is odd. The distribution of $(T^+(t),X^+(t))$ is given, for $x\ge 0$, by*

$$\mathbb{P}\{T^+(t)\in \mathrm{d}s,\ X^+(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=\frac{Ni}{2\pi}\sum_{m=0}^{\#J^+}\beta^+_{-m}(t-s)^{\frac{m-\#J^+}{N}}\int_0^\infty \xi^{m+\#K^+}\,e^{-s\xi^N}\,\mathcal{J}^+_m(x\xi)\,E_{1,\frac{m+\#K^+}{N}}\big(-(t-s)\xi^N\big)\,\mathrm{d}\xi \tag{6.5}$$

where

$$\mathcal{J}^+_m(z)=e^{-i\frac{\#J^+-m-1}{N}\pi}\sum_{j\in J^+}A^+_j(\theta^+_j)^{m+1}e^{-\theta^+_j e^{i\frac{\pi}{N}}z}-e^{i\frac{\#J^+-m-1}{N}\pi}\sum_{j\in J^+}A^+_j(\theta^+_j)^{m+1}e^{-\theta^+_j e^{-i\frac{\pi}{N}}z}.$$

Proof

When N is odd, we know that $(X^+)^*\stackrel{d}{=}X^-$ and then $((T^+)^*(t),(X^+)^*(t))\stackrel{d}{=}(T^-(t),X^-(t))$. Thus, by (6.1) and Lemma 6.1, for $x\ge 0$,

$$\mathbb{P}\{T^+(t)\in \mathrm{d}s,\ X^+(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=\mathbb{P}\{T^-(t)\in \mathrm{d}(t-s),\ X^-(t)\in \mathrm{d}(-x)\}/(\mathrm{d}s\,\mathrm{d}x)$$
$$=-\frac{Ni}{2\pi}\sum_{m=0}^{\#K^-}\alpha^-_{-m}(t-s)^{\frac{m-\#K^-}{N}}\int_0^\infty \xi^{m+\#J^-}\,e^{-s\xi^N}\,\mathcal{K}^-_m(-x\xi)\,E_{1,\frac{m+\#J^-}{N}}\big(-(t-s)\xi^N\big)\,\mathrm{d}\xi$$

where

$$\mathcal{K}^-_m(z)=e^{-i\frac{\#K^--m-1}{N}\pi}\sum_{k\in K^-}B^-_k(\theta^-_k)^{m+1}e^{-\theta^-_k e^{i\frac{\pi}{N}}z}-e^{i\frac{\#K^--m-1}{N}\pi}\sum_{k\in K^-}B^-_k(\theta^-_k)^{m+1}e^{-\theta^-_k e^{-i\frac{\pi}{N}}z}.$$

As in the proof of Theorem 6.2, we can write $\mathcal{K}^-_m(z)=(-1)^{m+1}\mathcal{J}^+_m(-z)$ where the function $\mathcal{J}^+_m$ is defined in Theorem 6.3. Finally, by replacing α^-_m by $(-1)^m\beta^+_m$ and $\#J^-$, $\#K^-$ by $\#K^+$, $\#J^+$ respectively, (6.5) ensues. ■

Formula (6.5) involves only quantities with associated '+' signs. We have a similar formula for X^- by changing all '+' into '−'. So, we can remove these signs in order to get a unified formula (this is (6.4)) which is valid for even N and, at least formally, for odd N without sign.

Remark 6.3. Let us integrate (6.4) with respect to x on $[0,\infty)$. We first calculate, recalling that $\mathcal{J}_m(z)=(-1)^{m+1}\mathcal{K}_m(-z)$ and referring to Remark 6.1,

$$\int_0^{\infty}\mathcal{J}_m(x\xi)\,\mathrm{d}x=(-1)^{m+1}\int_{-\infty}^{0}\mathcal{K}_m(x\xi)\,\mathrm{d}x=\begin{cases}0 & \text{if } 1\le m\le \#J,\\ -\dfrac{2i}{\xi}\sin\!\left(\dfrac{\#J}{N}\pi\right) & \text{if } m=0.\end{cases}$$

Then,

$$\begin{aligned}\mathbb{P}\{T(t)\in\mathrm{d}s,X(t)\ge 0\}/\mathrm{d}s&=\frac{N\sin\!\left(\frac{\#J}{N}\pi\right)}{\pi(t-s)^{\frac{\#J}{N}}}\int_0^{\infty}\xi^{\#K-1}\,e^{-s\xi^N}\,E_{1,\frac{\#K}{N}}\!\left(-(t-s)\xi^N\right)\mathrm{d}\xi\\&=\frac{N\sin\!\left(\frac{\#J}{N}\pi\right)}{\pi(t-s)^{\frac{\#J}{N}}}\sum_{\ell=0}^{\infty}\frac{(-(t-s))^{\ell}}{\Gamma\!\left(\ell+\frac{\#K}{N}\right)}\int_0^{\infty}\xi^{N\ell+\#K-1}e^{-s\xi^N}\,\mathrm{d}\xi\\&=\frac{\sin\!\left(\frac{\#J}{N}\pi\right)}{\pi s^{\frac{\#K}{N}}(t-s)^{\frac{\#J}{N}}}\sum_{\ell=0}^{\infty}\left(-\frac{t-s}{s}\right)^{\ell}=\frac{\sin\!\left(\frac{\#J}{N}\pi\right)}{\pi t}\left(\frac{s}{t-s}\right)^{\frac{\#J}{N}}.\end{aligned}$$

We retrieve (5.11).

6.3 Examples

In this part, we write out the distribution of the couple $(T(t),X(t))$ in the cases $N=2$, $N=3$ and $N=4$.

Example 6.1. Case $N=2$. Using the numerical values of Example 2.1, formula (6.1) yields, for $x\le 0$,

$$\begin{aligned}\mathbb{P}\{T(t)\in\mathrm{d}s,X(t)\in\mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)=\frac{1}{\pi}\Bigg(&\frac{1}{\sqrt{s}}\int_0^{\infty}\xi\,e^{-(t-s)\xi^2}\,\tilde{\mathcal{K}}_0(x\xi)\,E_{1,\frac12}\!\left(-s\xi^2\right)\mathrm{d}\xi\\&+\int_0^{\infty}\xi^2\,e^{-(t-s)\xi^2}\,\tilde{\mathcal{K}}_1(x\xi)\,E_{1,1}\!\left(-s\xi^2\right)\mathrm{d}\xi\Bigg)\end{aligned}$$

with

$$E_{1,1}\!\left(-s\xi^2\right)=e^{-s\xi^2},\quad \tilde{\mathcal{K}}_0(z)=-i\,\mathcal{K}_0(z)=i(e^{iz}-e^{-iz}),\quad \tilde{\mathcal{K}}_1(z)=-i\,\mathcal{K}_1(z)=e^{iz}+e^{-iz}.$$

On the other hand, we learn from Lemma 7.3 that, for $\xi\ge 0$,

$$E_{1,\frac12}(-s\xi^2)=\frac{1}{\sqrt{\pi}}\left(1-2\xi e^{-s\xi^2}\int_0^{s\xi}e^{\frac{u^2}{s}}\,\mathrm{d}u\right).$$

Therefore,

$$\begin{aligned}\mathbb{P}\{T(t)\in\mathrm{d}s,X(t)\in\mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)&=\frac{i}{\pi^{3/2}\sqrt{s}}\int_0^{\infty}\xi\left(e^{ix\xi-(t-s)\xi^2}-e^{-ix\xi-(t-s)\xi^2}\right)\mathrm{d}\xi\\&\quad-\frac{2i}{\pi^{3/2}\sqrt{s}}\int_0^{\infty}\xi^2\left(e^{ix\xi-t\xi^2}-e^{-ix\xi-t\xi^2}\right)\left(\int_0^{s\xi}e^{\frac{u^2}{s}}\,\mathrm{d}u\right)\mathrm{d}\xi\\&\quad+\frac{1}{\pi}\int_0^{\infty}\xi^2\left(e^{ix\xi-t\xi^2}+e^{-ix\xi-t\xi^2}\right)\mathrm{d}\xi\\&=\frac{i}{\pi^{3/2}\sqrt{s}}\int_{-\infty}^{\infty}\xi\,e^{ix\xi-(t-s)\xi^2}\mathrm{d}\xi+\frac{1}{\pi}\int_{-\infty}^{\infty}\xi^2e^{ix\xi-t\xi^2}\mathrm{d}\xi\\&\quad+\frac{4}{\pi^{3/2}\sqrt{s}}\int_0^{\infty}\xi^2\sin(x\xi)\,e^{-t\xi^2}\left(\int_0^{s\xi}e^{\frac{u^2}{s}}\,\mathrm{d}u\right)\mathrm{d}\xi.\end{aligned}$$

Let us compute two intermediate integrals:

$$\int_{-\infty}^{\infty} \xi\, e^{ix\xi-(t-s)\xi^2}\, \mathrm{d}\xi = e^{-\frac{x^2}{4(t-s)}} \int_{-\infty}^{\infty} \xi\, e^{-(t-s)\left(\xi-\frac{ix}{2(t-s)}\right)^2} \mathrm{d}\xi = e^{-\frac{x^2}{4(t-s)}} \int_{-\infty}^{\infty} \left(\xi+\frac{ix}{2(t-s)}\right) e^{-(t-s)\xi^2}\, \mathrm{d}\xi$$
$$= \frac{i\sqrt{\pi}\, x}{2(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}},$$

$$\int_{-\infty}^{\infty} \xi^2 e^{ix\xi-t\xi^2}\, \mathrm{d}\xi = e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} \left(\xi+\frac{ix}{2t}\right)^2 e^{-t\xi^2}\, \mathrm{d}\xi = e^{-\frac{x^2}{4t}} \left(\int_{-\infty}^{\infty} \xi^2 e^{-t\xi^2}\, \mathrm{d}\xi - \frac{x^2}{4t^2} \int_{-\infty}^{\infty} e^{-t\xi^2}\, \mathrm{d}\xi\right)$$
$$= \frac{\sqrt{\pi}}{2t^{3/2}}\, e^{-\frac{x^2}{4t}} \left(1-\frac{x^2}{2t}\right).$$

We deduce the following representation: for $x \le 0$,

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\, \mathrm{d}x) = \frac{4\sqrt{s}}{\pi^{3/2}} \left[\int_0^{\infty} \xi^2 \sin(x\xi)\, e^{-t\xi^2} \left(\int_0^{\xi} e^{su^2}\, \mathrm{d}u\right) \mathrm{d}\xi - \frac{\sqrt{\pi}}{8} \frac{xe^{-\frac{x^2}{4(t-s)}}}{s(t-s)^{3/2}}\right]$$
$$+ \frac{1}{2\sqrt{\pi}t^{3/2}} \left(1-\frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}}.$$

For $x \ge 0$, (6.4) gives

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\, \mathrm{d}x) = \frac{1}{\pi} \left(\frac{1}{\sqrt{t-s}} \int_0^{\infty} \xi\, e^{-s\xi^2}\, \tilde{\mathcal{J}}_0(x\xi)\, E_{1,\frac12}\left(-(t-s)\xi^2\right) \mathrm{d}\xi\right.$$
$$\left. + \int_0^{\infty} \xi^2\, e^{-s\xi^2}\, \tilde{\mathcal{J}}_1(x\xi)\, E_{1,1}\left(-(t-s)\xi^2\right) \mathrm{d}\xi\right)$$

with

$$\tilde{\mathcal{J}}_0(z) = i\, \mathcal{J}_0(z) = -i(e^{iz}-e^{-iz}), \quad \tilde{\mathcal{J}}_1(z) = -i\, \mathcal{J}_1(z) = e^{iz}+e^{-iz}.$$

As previously,

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\, \mathrm{d}x) = -\frac{i}{\pi^{3/2}\sqrt{t-s}} \int_{-\infty}^{\infty} \xi\, e^{ix\xi-s\xi^2} \mathrm{d}\xi + \frac{1}{\pi} \int_{-\infty}^{\infty} \xi^2 e^{ix\xi-t\xi^2} \mathrm{d}\xi$$
$$- \frac{4}{\pi^{3/2}\sqrt{t-s}} \int_0^{\infty} \xi^2 \sin(x\xi)\, e^{-t\xi^2} \left(\int_0^{(t-s)\xi} e^{\frac{u^2}{t-s}}\, \mathrm{d}u\right) \mathrm{d}\xi$$
$$= \frac{4\sqrt{t-s}}{\pi^{3/2}} \left[\frac{\sqrt{\pi}}{8} \frac{xe^{-\frac{x^2}{4s}}}{s^{3/2}(t-s)} - \int_0^{\infty} \xi^2 \sin(x\xi)\, e^{-t\xi^2} \left(\int_0^{\xi} e^{su^2}\, \mathrm{d}u\right) \mathrm{d}\xi\right]$$
$$+ \frac{1}{2\sqrt{\pi}t^{3/2}} \left(1-\frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}}.$$

Actually, the density of $(T(t), X(t))$ related to (rescaled) Brownian motion is well-known under another form. For instance, in [6] pages 129–131, we find that

$$\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\, \mathrm{d}x) = \begin{cases} \displaystyle\int_0^{\infty} \frac{y(y-x)}{4\pi s^{3/2}(t-s)^{3/2}}\, e^{-\frac{y^2}{4s}-\frac{(y-x)^2}{4(t-s)}}\, \mathrm{d}y & \text{if } x \le 0, \\ \displaystyle\int_0^{\infty} \frac{y(y+x)}{4\pi s^{3/2}(t-s)^{3/2}}\, e^{-\frac{(y+x)^2}{4s}-\frac{y^2}{4(t-s)}}\, \mathrm{d}y & \text{if } x \ge 0. \end{cases}$$

The coincidence of our representation and that of [6] can be checked by using Lemma 7.9 and Lemma 7.10 in the appendix.

Example 6.2. Case $N=3$.
• Suppose $\kappa_3 = 1$. Using $E_{1,1}\left(-s\xi^3\right) = e^{-s\xi^3}$ and the values of Example 2.2, (6.1) writes, for $x \le 0$,

$$
\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\qquad=\frac{\sqrt{3}}{2\pi}\bigg(s^{-2/3}\int_0^\infty \xi\, e^{-(t-s)\xi^3}\,\tilde{\mathcal{K}}_0(x\xi)\,E_{1,\frac13}\big(-s\xi^3\big)\,\mathrm{d}\xi\\
&\qquad\quad+s^{-1/3}\int_0^\infty \xi^2\, e^{-(t-s)\xi^3}\,\tilde{\mathcal{K}}_1(x\xi)\,E_{1,\frac23}\big(-s\xi^3\big)\,\mathrm{d}\xi+\int_0^\infty \xi^3\, e^{-t\xi^3}\,\tilde{\mathcal{K}}_2(x\xi)\,\mathrm{d}\xi\bigg)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{K}}_0(z)&=-i\sqrt{3}\,\mathcal{K}_0(z)=e^z-e^{-z/2}\left(\cos\frac{\sqrt{3}\,z}{2}+\sqrt{3}\,\sin\frac{\sqrt{3}\,z}{2}\right),\\
\tilde{\mathcal{K}}_1(z)&=-i\sqrt{3}\,\mathcal{K}_1(z)=-e^z+e^{-z/2}\left(\cos\frac{\sqrt{3}\,z}{2}-\sqrt{3}\,\sin\frac{\sqrt{3}\,z}{2}\right),\\
\tilde{\mathcal{K}}_2(z)&=-i\sqrt{3}\,\mathcal{K}_2(z)=e^z+2e^{-z/2}\cos\frac{\sqrt{3}\,z}{2}.
\end{aligned}
$$

For $x\geq 0$, (6.5) gives

$$
\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\qquad=\frac{3}{2\pi}\left(\frac{1}{\sqrt[3]{t-s}}\int_0^\infty \xi^2\, e^{-s\xi^3}\,\tilde{\mathcal{J}}_0(x\xi)\,E_{1,\frac23}\big(-(t-s)\xi^3\big)\,\mathrm{d}\xi+\int_0^\infty \xi^3\, e^{-t\xi^3}\,\tilde{\mathcal{J}}_1(x\xi)\,\mathrm{d}\xi\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{J}}_0(z)&=i\,\mathcal{J}_0(z)=2\,e^{-z/2}\,\sin\frac{\sqrt{3}\,z}{2},\\
\tilde{\mathcal{J}}_1(z)&=-i\,\mathcal{J}_1(z)=\;e^{-z/2}\left(\sqrt{3}\,\cos\frac{\sqrt{3}\,z}{2}-\sin\frac{\sqrt{3}\,z}{2}\right).
\end{aligned}
$$

• Suppose $\kappa_3=-1$. Likewise, for $x\leq 0$,

$$
\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\qquad=\frac{3}{2\pi}\left(\frac{1}{\sqrt[3]{s}}\int_0^\infty \xi^2\, e^{-(t-s)\xi^3}\,\tilde{\mathcal{K}}_0(x\xi)\,E_{1,\frac23}\big(-s\xi^3\big)\,\mathrm{d}\xi\;+\int_0^\infty \xi^3\, e^{-t\xi^3}\tilde{\mathcal{K}}_1(x\xi)\,\mathrm{d}\xi\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{K}}_0(z)&=-i\,\mathcal{K}_0(z)=-2\,e^{z/2}\,\sin\frac{\sqrt{3}\,z}{2},\\
\tilde{\mathcal{K}}_1(z)&=-i\,\mathcal{K}_1(z)=e^{z/2}\left(\sqrt{3}\,\cos\frac{\sqrt{3}\,z}{2}+\sin\frac{\sqrt{3}\,z}{2}\right).
\end{aligned}
$$

For $x\geq 0$,

$$
\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\qquad=\frac{\sqrt{3}}{2\pi}\bigg((t-s)^{-2/3}\int_0^\infty \xi\, e^{-s\xi^3}\,\tilde{\mathcal{J}}_0(x\xi)\,E_{1,\frac13}\big(-(t-s)\xi^3\big)\,\mathrm{d}\xi\\
&\qquad\quad+(t-s)^{-1/3}\int_0^\infty \xi^2\, e^{-s\xi^3}\,\tilde{\mathcal{J}}_1(x\xi)\,E_{1,\frac23}\big(-(t-s)\xi^3\big)\,\mathrm{d}\xi+\int_0^\infty \xi^3\, e^{-t\xi^3}\,\tilde{\mathcal{J}}_2(x\xi)\,\mathrm{d}\xi\bigg)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{J}}_0(z)&=i\sqrt{3}\,\mathcal{J}_0(z)=e^{-z}-e^{z/2}\left(\cos\frac{\sqrt{3}\,z}{2}-\sqrt{3}\,\sin\frac{\sqrt{3}\,z}{2}\right),\\
\tilde{\mathcal{J}}_1(z)&=-i\sqrt{3}\,\mathcal{J}_1(z)=-e^{-z}+e^{z/2}\left(\cos\frac{\sqrt{3}\,z}{2}+\sqrt{3}\,\sin\frac{\sqrt{3}\,z}{2}\right),\\
\tilde{\mathcal{J}}_2(z)&=i\sqrt{3}\,\mathcal{J}_2(z)=e^{-z}+2\,e^{z/2}\cos\frac{\sqrt{3}\,z}{2}.
\end{aligned}
$$

Example 6.3. Case $N = 4$. Referring to Example 2.3, formula (6.1) writes, for $x \le 0$,

$$\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\quad = \frac{2}{\pi}\left(\frac{1}{\sqrt{s}}\int_0^\infty \xi^2\, e^{-(t-s)\xi^4}\,\tilde{\mathcal{K}}_0(x\xi)\,E_{1,\frac12}\big(-s\xi^4\big)\,\mathrm{d}\xi\right.\\
&\qquad \left. +\frac{\sqrt{2}}{\sqrt[4]{s}}\int_0^\infty \xi^3\, e^{-(t-s)\xi^4}\,\tilde{\mathcal{K}}_1(x\xi)\,E_{1,\frac34}\big(-s\xi^4\big)\,\mathrm{d}\xi+\int_0^\infty \xi^4\, e^{-t\xi^4}\,\tilde{\mathcal{K}}_2(x\xi)\,\mathrm{d}\xi\right)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{K}}_0(z) &= -i\,\mathcal{K}_0(z) = e^z-\cos z-\sin z,\\
\tilde{\mathcal{K}}_1(z) &= -i\,\mathcal{K}_1(z) = -e^z+\cos z-\sin z,\\
\tilde{\mathcal{K}}_2(z) &= -i\,\mathcal{K}_2(z) = e^z+\cos z+\sin z.
\end{aligned}$$

For $x \ge 0$, (6.4) reads

$$\begin{aligned}
&\mathbb{P}\{T(t)\in \mathrm{d}s, X(t)\in \mathrm{d}x\}/(\mathrm{d}s\,\mathrm{d}x)\\
&\quad = \frac{2}{\pi}\left(\frac{1}{\sqrt{t-s}}\int_0^\infty \xi^2\, e^{-s\xi^4}\,\tilde{\mathcal{J}}_0(x\xi)\,E_{1,\frac12}\big(-(t-s)\xi^4\big)\,\mathrm{d}\xi\right.\\
&\qquad \left. +\frac{\sqrt{2}}{\sqrt[4]{t-s}}\int_0^\infty \xi^3\, e^{-s\xi^4}\,\tilde{\mathcal{J}}_1(x\xi)\,E_{1,\frac34}\big(-(t-s)\xi^4\big)\,\mathrm{d}\xi+\int_0^\infty \xi^4\, e^{-t\xi^4}\,\tilde{\mathcal{J}}_2(x\xi)\,\mathrm{d}\xi\right)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{J}}_0(z) &= i\,\mathcal{J}_0(z) = e^{-z}-\cos z+\sin z,\\
\tilde{\mathcal{J}}_1(z) &= -i\,\mathcal{J}_1(z) = -e^{-z}+\cos z+\sin z,\\
\tilde{\mathcal{J}}_2(z) &= i\,\mathcal{J}_2(z) = e^{-z}+\cos z-\sin z.
\end{aligned}$$

7 Appendix

Lemma 7.1 (Spitzer)**.** *Let $(\xi_k)_{k\ge1}$ be a sequence of independent identically distributed random variables and set $X_0 = 0$ and $T_0 = 0$ and, for any $k \ge 1$,*

$$X_k = \xi_1+\cdots+\xi_k, \qquad T_k = \sum_{j=1}^{k} \mathbb{1}_{[0,+\infty)}(X_k).$$

Then, for $\mu\in\mathbb{R}$, $\nu>0$ and $|z|<1$,

$$\sum_{k=0}^{\infty}\mathbb{E}\big[e^{i\mu X_k-\nu T_k}\big]\,z^k = \exp\!\left(\sum_{k=1}^{\infty}\mathbb{E}\Big[e^{i\mu X_k-\nu k\mathbb{1}_{[0,+\infty)}(X_k)}\Big]\,\frac{z^k}{k}\right), \tag{7.1}$$

$$\sum_{k=0}^{\infty}\mathbb{E}\big[e^{i\mu X_k-\nu T_k}\mathbb{1}_{[0,+\infty)}(X_k)\big]\,z^k = \frac{1}{e^\nu-1}\left[e^\nu-\exp\!\left(-\sum_{k=1}^{\infty}\big(1-e^{-\nu k}\big)\,\mathbb{E}\big[e^{i\mu X_k}\mathbb{1}_{[0,+\infty)}(X_k)\big]\,\frac{z^k}{k}\right)\right], \tag{7.2}$$

$$\sum_{k=0}^{\infty}\mathbb{E}\big[e^{i\mu X_k-\nu T_k}\mathbb{1}_{(-\infty,0)}(X_k)\big]\,z^k = \frac{e^\nu}{e^\nu-1}\left[\exp\!\left(\sum_{k=1}^{\infty}\big(1-e^{-\nu k}\big)\,\mathbb{E}\big[e^{i\mu X_k}\mathbb{1}_{(-\infty,0)}(X_k)\big]\,\frac{z^k}{k}\right)-1\right]. \tag{7.3}$$

Proof
Formula (7.1) is stated in [21] without proof. So, we produce a proof below which is rather similar to one lying in [21] related to the maximum functional of the X_k's.

• **Step 1.** Set, for any $(x_1,\dots,x_n)\in\mathbb{R}^n$ and $\sigma\in\mathfrak{S}_n$ ($\mathfrak{S}_n$ being the set of the permutations of $1,2,\dots,n$),

$$U(x_1,\dots,x_n) = \sum_{k=1}^{n}\mathbb{1}_{[0,\infty)}\left(\sum_{j=1}^{k}x_j\right)$$

and

$$V(\sigma; x_1, \dots, x_n) = \sum_{k=1}^{n_\sigma} \# c_k(\sigma) \mathbb{1}_{[0,\infty)}\Bigg(\sum_{j \in c_k(\sigma)} x_j\Bigg).$$

In the definition of V above, the permutation σ is decomposed into n_σ cycles: $\sigma = (c_1(\sigma))(c_2(\sigma)) \dots (c_{n_\sigma}(\sigma))$.

In view of Theorem 2.3 in [21], we have the equality between the two following sets:

$$\{U(\sigma(x_1), \dots, \sigma(x_n)), \sigma \in \mathfrak{S}_n\} = \{V(\sigma; x_1, \dots, x_n), \sigma \in \mathfrak{S}_n\}.$$

We then deduce, for any bounded Borel functions ϕ and F,

$$\mathbb{E}[\phi(X_n) F(U(\xi_1, \dots, \xi_n))] = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbb{E}\Bigg[\phi\Bigg(\sum_{j=1}^{n} \xi_{\sigma(j)}\Bigg) F(V(\sigma; \xi_1, \dots, \xi_n))\Bigg].$$

In particular, for $\phi(x) = e^{i\mu x}$ and $F(x) = e^{-\nu x}$ (where $\mu \in \mathbb{R}$ and $\nu > 0$ are fixed),

$$\begin{aligned}
\mathbb{E}\Big[e^{i\mu X_n - \nu \sum_{k=1}^{n} \mathbb{1}_{[0,+\infty)}(\sum_{j=1}^{k} \xi_j)}\Big] &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbb{E}\Bigg[\exp\Bigg(i\mu \sum_{k=1}^{n_\sigma} \sum_{j \in c_k(\sigma)} \xi_j - \nu \sum_{k=1}^{n_\sigma} \# c_k(\sigma) \mathbb{1}_{[0,\infty)}\Bigg(\sum_{j \in c_k(\sigma)} \xi_j\Bigg)\Bigg)\Bigg] \\
&= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{k=1}^{n_\sigma} \mathbb{E}\Bigg[\exp\Bigg(i\mu \sum_{j \in c_k(\sigma)} \xi_j - \nu\, (\# c_k(\sigma)) \mathbb{1}_{[0,\infty)}\Bigg(\sum_{j \in c_k(\sigma)} \xi_j\Bigg)\Bigg)\Bigg] \\
&= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{k=1}^{n_\sigma} \mathbb{E}\Bigg[\exp\Bigg(i\mu \sum_{j=1}^{\# c_k(\sigma)} \xi_j - \nu\, (\# c_k(\sigma)) \mathbb{1}_{[0,\infty)}\Bigg(\sum_{j=1}^{\# c_k(\sigma)} \xi_j\Bigg)\Bigg)\Bigg].
\end{aligned}$$

Denote by $r_\ell(\sigma)$ the number of cycles of length ℓ in σ for any $\ell \in \{1, \dots, n\}$. We have $r_1(\sigma) + 2r_2(\sigma) + \dots + nr_n(\sigma) = n$. Then,

$$\begin{aligned}
\mathbb{E}\big[e^{i\mu X_n - \nu T_n}\big] &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{\ell=1}^{n} \Big(\mathbb{E}\Big[e^{i\mu X_l - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\Big]\Big)^{r_\ell(\sigma)} \\
&= \frac{1}{n!} \sum_{\substack{k_1, \dots, k_n \geq 0: \\ k_1 + 2k_2 + \dots + nk_n = n}} N_{k_1, \dots, k_n} \prod_{\ell=1}^{n} \Big(\mathbb{E}\Big[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\Big]\Big)^{k_\ell}
\end{aligned}$$

where $N_{k_1, \dots, k_n}$ is the number of the permutations σ of n objects satisfying $r_1(\sigma) = k_1, \dots, r_n(\sigma) = k_n$; this number is equal to

$$N_{k_1, \dots, k_n} = \frac{n!}{(k_1! 1^{k_1})(k_2! 2^{k_2}) \dots (k_n! n^{k_n})}.$$

Then,

$$\mathbb{E}\big[e^{i\mu X_n - \nu T_n}\big] = \sum_{\substack{k_1, \dots, k_n \geq 0: \\ k_1 + 2k_2 + \dots + nk_n = n}} \prod_{\ell=1}^{n} \frac{1}{k_\ell! \ell^{k_\ell}} \left(\mathbb{E}\left[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\right]\right)^{k_\ell}.$$

• **Step 2.** Therefore, the identity between the generating functions follows: for $|z| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}\big[e^{i\mu X_n - \nu T_n}\big] z^n &= \sum_{\substack{n \geq 0, k_1, \dots, k_n \geq 0: \\ k_1 + 2k_2 + \dots + nk_n = n}} \prod_{\ell=1}^{n} \frac{1}{k_\ell!} \left(\mathbb{E}\Big[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\Big] \frac{z^\ell}{\ell}\right)^{k_\ell} \\
&= \sum_{k_1, k_2, \dots \geq 0} \prod_{\ell=1}^{\infty} \frac{1}{k_\ell!} \left(\mathbb{E}\left[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\right] \frac{z^\ell}{\ell}\right)^{k_\ell} \\
&= \prod_{\ell=1}^{\infty} \left[\sum_{k=1}^{\infty} \frac{1}{k!} \left(\mathbb{E}\left[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\right] \frac{z^\ell}{\ell}\right)^{k}\right] \\
&= \prod_{\ell=1}^{\infty} \exp\left(\mathbb{E}\Big[e^{i\mu X_\ell - \nu \ell\, \mathbb{1}_{[0,\infty)}(X_\ell)}\Big] \frac{z^\ell}{\ell}\right)
\end{aligned}$$

$$= \exp\left(\sum_{n=1}^{\infty} \mathbb{E}\left[e^{i\mu X_n - \nu n \mathbb{1}_{[0,+\infty)}(X_n)}\right] \frac{z^n}{n}\right).$$

The proof of (7.1) is finished.

• **Step 3.**

Using the elementary identity $e^{a\mathbb{1}_A(x)} - 1 = (e^a - 1)\mathbb{1}_A(x)$ and noticing that $T_k = T_{k-1} + \mathbb{1}_{[0,+\infty)}(X_k)$, we get for any $k \geq 1$,

$$\mathbb{E}\big[e^{i\mu X_k - \nu T_k} \mathbb{1}_{[0,+\infty)}(X_k)\big] = \mathbb{E}\left[e^{i\mu X_k - \nu T_k} \frac{e^{\nu \mathbb{1}_{[0,+\infty)}(X_k)} - 1}{e^\nu - 1}\right] = \frac{1}{e^\nu - 1}\left[\mathbb{E}\big(e^{i\mu X_k - \nu T_{k-1}}\big) - \mathbb{E}\big(e^{i\mu X_k - \nu T_k}\big)\right].$$

Now, since $X_k = X_{k-1} + \xi_k$ where X_{k-1} and ξ_k are independent and ξ_k have the same distribution as ξ_1, we have, for $k \geq 1$,

$$\mathbb{E}\big(e^{i\mu X_k - \nu T_{k-1}}\big) = \mathbb{E}\big(e^{i\mu \xi_1}\big)\,\mathbb{E}\big(e^{i\mu X_{k-1} - \nu T_{k-1}}\big).$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{E}\big[e^{i\mu X_k - \nu T_k} \mathbb{1}_{[0,+\infty)}(X_k)\big]\, z^k &= \frac{1}{e^\nu - 1} \sum_{k=1}^{\infty} \left(\mathbb{E}\big[e^{i\mu X_k - \nu T_{k-1}}\big] - \mathbb{E}\big[e^{i\mu X_k - \nu T_k}\big]\right) z^k \\
&= \frac{1}{e^\nu - 1}\left(\mathbb{E}\big(e^{i\mu\xi_1}\big) \sum_{k=1}^{\infty} \mathbb{E}\big[e^{i\mu X_{k-1} - \nu T_{k-1}}\big]\, z^k - \sum_{k=1}^{\infty} \mathbb{E}\big[e^{i\mu X_k - \nu T_k}\big]\, z^k\right) \\
&= \frac{1}{e^\nu - 1}\left(\big(z\,\mathbb{E}\big(e^{i\mu\xi_1}\big) - 1\big) \sum_{k=0}^{\infty} \mathbb{E}\big[e^{i\mu X_k - \nu T_k}\big]\, z^k + 1\right). \qquad (7.4)
\end{aligned}$$

By putting (7.1) into (7.4), we extract

$$\sum_{k=0}^{\infty} \mathbb{E}\big[e^{i\mu X_k - \nu T_k} \mathbb{1}_{[0,+\infty)}(X_k)\big]\, z^k = \frac{1}{e^\nu - 1}\left[e^\nu - \big(1 - z\,\mathbb{E}\big(e^{i\mu\xi_1}\big)\big)\, S(\mu,\nu,z)\right] \qquad (7.5)$$

where we set

$$S(\mu,\nu,z) = \exp\left(\sum_{k=1}^{\infty} \mathbb{E}\left[e^{i\mu X_k - \nu k \mathbb{1}_{[0,+\infty)}(X_k)}\right] \frac{z^k}{k}\right).$$

Next, using the elementary identity $1 - \zeta = \exp[\log(1-\zeta)] = \exp\big[-\sum_{k=1}^{\infty} \zeta^k/k\big]$ valid for $|\zeta| < 1$,

$$1 - z\,\mathbb{E}\big(e^{i\mu\xi_1}\big) = \exp\left(-\sum_{k=1}^{\infty} \big[\mathbb{E}\big(e^{i\mu\xi_1}\big)\big]^k \frac{z^k}{k}\right) = \exp\left(-\sum_{k=1}^{\infty} \mathbb{E}\big(e^{i\mu X_k}\big) \frac{z^k}{k}\right)$$

and then

$$\begin{aligned}
\big(1 - z\,\mathbb{E}\big(e^{i\mu\xi_1}\big)\big)\, S(\mu,\nu,z) &= \exp\left(\sum_{k=1}^{\infty} \mathbb{E}\left[e^{i\mu X_k - \nu k \mathbb{1}_{[0,+\infty)}(X_k)} - e^{i\mu X_k}\right] \frac{z^k}{k}\right) \\
&= \exp\left(-\sum_{k=1}^{\infty} \big(1 - e^{-\nu k}\big)\, \mathbb{E}\big[e^{i\mu X_k} \mathbb{1}_{[0,+\infty)}(X_k)\big] \frac{z^k}{k}\right). \qquad (7.6)
\end{aligned}$$

Hence, by putting (7.6) into (7.5), formula (7.2) entails.

By subtracting (7.5) from (7.1), we obtain the intermediate representation

$$\sum_{k=0}^{\infty} \mathbb{E}\big[e^{i\mu X_k - \nu T_k} \mathbb{1}_{(-\infty,0)}(X_k)\big]\, z^k = \frac{1}{e^\nu - 1}\left[\big(e^\nu - z\,\mathbb{E}\big(e^{i\mu\xi_1}\big)\big)\, S(\mu,\nu,z) - e^\nu\right].$$

By writing, as previously,

$$e^\nu - z\,\mathbb{E}\big(e^{i\mu\xi_1}\big) = e^\nu \exp\left(-\sum_{k=1}^{\infty} \mathbb{E}\big(e^{i\mu X_k}\big) \frac{e^{-\nu k} z^k}{k}\right),$$

we find

$$\big(e^{\nu}-z\,\mathbb{E}\big(e^{i\mu\xi_1}\big)\big)\,S(\mu,\nu,z)=e^{\nu}\exp\!\left(\sum_{k=1}^{\infty}\mathbb{E}\Big[e^{i\mu X_k-\nu k\mathbb{1}_{[0,+\infty)}(X_k)}-e^{i\mu X_k-\nu k}\Big]\,\frac{z^k}{k}\right)$$
$$=e^{\nu}\exp\!\left(\sum_{k=1}^{\infty}\big(1-e^{-\nu k}\big)\,\mathbb{E}\big[e^{i\mu X_k}\mathbb{1}_{(-\infty,0)}(X_k)\big]\,\frac{z^k}{k}\right).$$

Finally, (7.3) ensues. ■

Lemma 7.2. *The following identities hold:*

$$\beta_{\#K}=(-1)^{\#K-1}\prod_{k\in K}\theta_k,\qquad \beta_{\#K+1}=(-1)^{\#K-1}\left(\prod_{k\in K}\theta_k\right)\!\left(\sum_{k\in K}\theta_k\right).$$

Proof

We label the set K as $\{1,2,3,\dots,\#K\}$. By (2.5), we know that the B_k's solve a Vandermonde system. Then, by Cramer's formulas, we can write them as fractions of some determinants: $B_k=V_k/V$ where

$$V=\begin{vmatrix}1&\dots&1\\ \theta_1&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots\\ \theta_1^{\#K-1}&\dots&\theta_{\#K}^{\#K-1}\end{vmatrix}\quad\text{and}\quad V_k=\begin{vmatrix}1&\dots&1&1&1&\dots&1\\ \theta_1&\dots&\theta_{k-1}&0&\theta_{k+1}&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{k-1}^2&0&\theta_{k+1}^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots&\vdots&\vdots&&\vdots\\ \theta_1^{\#K-1}&\dots&\theta_{k-1}^{\#K-1}&0&\theta_{k+1}^{\#K-1}&\dots&\theta_{\#K}^{\#K-1}\end{vmatrix}.$$

By expanding the determinant V_k with respect to its k^{th} column and next factorizing it suitably, we easily see that

$$V_k=(-1)^{k+1}\begin{vmatrix}\theta_1&\dots&\theta_{k-1}&\theta_{k+1}&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{k-1}^2&\theta_{k+1}^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots&\vdots&&\vdots\\ \theta_1^{\#K-1}&\dots&\theta_{k-1}^{\#K-1}&\theta_{k+1}^{\#K-1}&\dots&\theta_{\#K}^{\#K-1}\end{vmatrix}$$
$$=(-1)^{k+1}\frac{\prod_{i\in K}\theta_i}{\theta_k}\begin{vmatrix}1&\dots&1&1&\dots&1\\ \theta_1&\dots&\theta_{k-1}&\theta_{k+1}&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{k-1}^2&\theta_{k+1}^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots&\vdots&&\vdots\\ \theta_1^{\#K-2}&\dots&\theta_{k-1}^{\#K-2}&\theta_{k+1}^{\#K-2}&\dots&\theta_{\#K}^{\#K-2}\end{vmatrix}.$$

With this at hands, we have

$$\beta_{\#K}=\sum_{k\in K}B_k\theta_k^{\#K}=\frac{\prod_{k\in K}\theta_k}{V}\sum_{k\in K}(-1)^{k+1}\theta_k^{\#K-1}\begin{vmatrix}1&\dots&1&1&\dots&1\\ \theta_1&\dots&\theta_{k-1}&\theta_{k+1}&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{k-1}^2&\theta_{k+1}^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots&\vdots&&\vdots\\ \theta_1^{\#K-2}&\dots&\theta_{k-1}^{\#K-2}&\theta_{k+1}^{\#K-2}&\dots&\theta_{\#K}^{\#K-2}\end{vmatrix}.$$

We can observe that the sum lying on the above right-hand side is nothing but the expansion of the determinant V with respect to its last row multiplied by the sign $(-1)^{\#K-1}$. This immediately ensues that $\beta_{\#K}=(-1)^{\#K-1}\prod_{k\in K}\theta_k$. Similarly,

$$\beta_{\#K+1}=\sum_{k\in K}B_k\theta_k^{\#K+1}=\frac{\prod_{k\in K}\theta_k}{V}\sum_{k\in K}(-1)^{k+1}\theta_k^{\#K}\begin{vmatrix}1&\dots&1&1&\dots&1\\ \theta_1&\dots&\theta_{k-1}&\theta_{k+1}&\dots&\theta_{\#K}\\ \theta_1^2&\dots&\theta_{k-1}^2&\theta_{k+1}^2&\dots&\theta_{\#K}^2\\ \vdots&&\vdots&\vdots&&\vdots\\ \theta_1^{\#K-2}&\dots&\theta_{k-1}^{\#K-2}&\theta_{k+1}^{\#K-2}&\dots&\theta_{\#K}^{\#K-2}\end{vmatrix}.$$

The above sum is the expansion with respect to its last row, multiplied by the sign $(-1)^{\#K-1}$, of the determinant V' defined as

$$V' = \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{\#K} \\ \theta_1^2 & \dots & \theta_{\#K}^2 \\ \vdots & & \vdots \\ \theta_1^{\#K-2} & \dots & \theta_{\#K}^{\#K-2} \\ \theta_1^{\#K} & \dots & \theta_{\#K}^{\#K} \end{vmatrix}.$$

Let $R_0, R_1, R_2, \dots, R_{\#K-2}, R_{\#K-1}$ denote the rows of V'. We perform the substitution $R_{\#K-1} \leftarrow R_{\#K-1} + \sum_{\ell=2}^{\#K}(-1)^\ell \sigma_\ell R_{\#K-\ell}$ where the σ_ℓ's are defined by (2.3). This substitution does not affect the value of V' and it transforms, e.g., the first term of the last row into

$$\theta_1^{\#K} + \sum_{\ell=2}^{\#K}(-1)^\ell \sigma_\ell \, \theta_1^{\#K-\ell}.$$

Recall that $\sigma_\ell = \sum_{1\le k_1<\dots<k_\ell\le \#K} \theta_{k_1}\dots\theta_{k_\ell}$. We decompose σ_ℓ, by isolating the terms involving θ_1, into

$$\theta_1 \sum_{2\le k_2<\dots<k_\ell\le \#K} \theta_{k_2}\dots\theta_{k_\ell} + \sum_{2\le k_1<k_2<\dots<k_\ell\le \#K} \theta_{k_1}\theta_{k_2}\dots\theta_{k_\ell} = \theta_1\, \sigma'_{\ell-1} + \sigma'_\ell$$

where we set $\sigma'_{\#K} = 0$ and $\sigma'_\ell = \sum_{2\le k_1<k_2<\dots<k_\ell\le \#K} \theta_{k_1}\theta_{k_2}\dots\theta_{k_\ell}$. Therefore, we have

$$\begin{aligned} \theta_1^{\#K} + \sum_{\ell=2}^{\#K}(-1)^\ell \sigma_\ell \, \theta_1^{\#K-\ell} &= \theta_1^{\#K} + \sum_{\ell=2}^{\#K}(-1)^\ell \sigma'_{\ell-1} \, \theta_1^{\#K-\ell+1} + \sum_{\ell=2}^{\#K}(-1)^\ell \sigma'_\ell \, \theta_1^{\#K-\ell} \\ &= \theta_1^{\#K} + \sigma'_1 \, \theta_1^{\#K-1} = \theta_1^{\#K-1}(\theta_1 + \sigma'_1) = \theta_1^{\#K-1}\left(\sum_{k\in K}\theta_k\right). \end{aligned}$$

The foregoing manipulation works similarly for each term of the last row of V'. So, we deduce that $V' = \left(\sum_{k\in K}\theta_k\right) V$ and finally $\beta_{\#K+1} = (-1)^{\#K-1}\left(\prod_{k\in K}\theta_k\right)\left(\sum_{k\in K}\theta_k\right)$. ■

Lemma 7.3. *For $\alpha > 0$, the Mittag-Leffler functions $E_{1,\alpha}$ and $E_{1,\alpha+1}$ admit the following integral representations:*

$$E_{1,\alpha}(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\left(1 + x^{1-\alpha}e^x \displaystyle\int_0^x u^{\alpha-1}e^{-u}\,\mathrm{d}u\right) & \text{if } x > 0, \\ \dfrac{1}{\Gamma(\alpha)}\left(1 - |x|^{1-\alpha}e^x \displaystyle\int_0^{|x|} u^{\alpha-1}e^{u}\,\mathrm{d}u\right) & \text{if } x < 0, \end{cases} \tag{7.7}$$

$$E_{1,\alpha+1}(x) = \begin{cases} \dfrac{e^x}{\Gamma(\alpha)\,x^\alpha}\displaystyle\int_0^x u^{\alpha-1}e^{-u}\,\mathrm{d}u & \text{if } x > 0, \\ \dfrac{e^x}{\Gamma(\alpha)\,|x|^\alpha}\displaystyle\int_0^{|x|} u^{\alpha-1}e^{u}\,\mathrm{d}u & \text{if } x < 0. \end{cases} \tag{7.8}$$

Proof

Using the series expansion of $E_{1,\alpha}$, we obtain

$$\begin{aligned} xE'_{1,\alpha}(x) &= \sum_{r=1}^\infty \frac{rx^r}{\Gamma(r+\alpha)} = \sum_{r=1}^\infty \frac{(r+\alpha-1)x^r}{\Gamma(r+\alpha)} + (1-\alpha)\sum_{r=1}^\infty \frac{x^r}{\Gamma(r+\alpha)} \\ &= x\sum_{r=1}^\infty \frac{x^{r-1}}{\Gamma(r+\alpha-1)} + (1-\alpha)\left(E_{1,\alpha}(x) - \frac{1}{\Gamma(\alpha)}\right) = (x+1-\alpha)\,E_{1,\alpha}(x) + \frac{\alpha-1}{\Gamma(\alpha)}. \end{aligned}$$

Hence, $E_{1,\alpha}$ solves the differential equation $xE'_{1,\alpha}(x) = (x+1-\alpha)E_{1,\alpha}(x) + \frac{\alpha-1}{\Gamma(\alpha)}$. In view of this equation, we know that $E_{1,\alpha}(x)$ is of the form

$$E_{1,\alpha}(x) = \lambda(x)\, e^{\int \frac{x+1-\alpha}{x}\,\mathrm{d}x} = \lambda(x)\,|x|^{1-\alpha}\,e^x$$

where the unknown function λ solves

$$\lambda'(x) = \frac{\alpha-1}{\Gamma(\alpha)x}|x|^{\alpha-1}e^{-x} = \begin{cases} \dfrac{\alpha-1}{\Gamma(\alpha)}\, x^{\alpha-2}\, e^{-x} & \text{if } x>0, \\ -\dfrac{\alpha-1}{\Gamma(\alpha)}\, |x|^{\alpha-2}\, e^{-x} & \text{if } x<0. \end{cases}$$

This implies that, for a certain $x_0 > 0$ and $\lambda_0 \in \mathbb{R}$ (λ_0 could be different for $x > 0$ and $x < 0$), we have, for $x > 0$,

$$\lambda(x) = \frac{\alpha-1}{\Gamma(\alpha)}\int_{x_0}^{x} u^{\alpha-2}\, e^{-u}\,\mathrm{d}u + \lambda_0 = \frac{1}{\Gamma(\alpha)}\left(x^{\alpha-1}\, e^{-x} - x_0^{\alpha-1}\, e^{-x_0} + \int_{x_0}^{x} u^{\alpha-1}\, e^{-u}\,\mathrm{d}u\right) + \lambda_0,$$

and, for $x < 0$,

$$\lambda(x) = \frac{\alpha-1}{\Gamma(\alpha)}\int_{x_0}^{|x|} u^{\alpha-2}e^{u}\,\mathrm{d}u + \lambda_0 = \frac{1}{\Gamma(\alpha)}\left(|x|^{\alpha-1}\, e^{|x|} - x_0^{\alpha-1}\, e^{x_0} - \int_{x_0}^{|x|} u^{\alpha-1}\, e^{u}\,\mathrm{d}u\right) + \lambda_0.$$

Then

$$E_{1,\alpha}(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} + \left(\lambda_0 - \dfrac{e^{-x_0}x_0^{\alpha-1}}{\Gamma(\alpha)}\right)x^{1-\alpha}e^{x} + \dfrac{x^{1-\alpha}e^{x}}{\Gamma(\alpha)}\displaystyle\int_{x_0}^{x} u^{\alpha-1}e^{-u}\,\mathrm{d}u & \text{if } x>0, \\ \dfrac{1}{\Gamma(\alpha)} + \left(\lambda_0 - \dfrac{e^{x_0}x_0^{\alpha-1}}{\Gamma(\alpha)}\right)|x|^{1-\alpha}e^{x} - \dfrac{|x|^{1-\alpha}e^{x}}{\Gamma(\alpha)}\displaystyle\int_{x_0}^{|x|} u^{\alpha-1}e^{u}\,\mathrm{d}u & \text{if } x<0. \end{cases} \tag{7.9}$$

Because of the singularity of the differential equation at zero, the initial value at zero does not determine x_0 and λ_0. Nevertheless, we know that $E_{1,\alpha}$ is C^1 at zero. So, we need to compute $E'_{1,\alpha}(x)$ for $x \neq 0$:

$$E'_{1,\alpha}(x) = \begin{cases} \left(\lambda_0 - \dfrac{x_0^{\alpha-1}e^{-x_0}}{\Gamma(\alpha)} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{x_0}^{x} u^{\alpha-1}e^{-u}\,\mathrm{d}u\right)\left(x^{1-\alpha}e^{x} + (1-\alpha)x^{-\alpha}e^{x}\right) + \dfrac{1}{\Gamma(\alpha)} & \text{if } x>0, \\ \left(\lambda_0 - \dfrac{x_0^{\alpha-1}e^{x_0}}{\Gamma(\alpha)} - \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{x_0}^{|x|} u^{\alpha-1}e^{u}\,\mathrm{d}u\right)\left(|x|^{1-\alpha}e^{x} - (1-\alpha)|x|^{-\alpha}e^{x}\right) + \dfrac{1}{\Gamma(\alpha)} & \text{if } x<0. \end{cases}$$

In order to have a C^1-function at 0, we must have

$$\lambda_0 - \frac{x_0^{\alpha-1}e^{-x_0}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_0^{x_0} u^{\alpha-1}e^{-u}\,\mathrm{d}u \quad \text{and} \quad \lambda_0 - \frac{x_0^{\alpha-1}e^{x_0}}{\Gamma(\alpha)} = -\frac{1}{\Gamma(\alpha)}\int_0^{x_0} u^{\alpha-1}e^{u}\,\mathrm{d}u. \tag{7.10}$$

Putting (7.10) into (7.9) yields (7.7). Next, formula (7.8) can be deduced from (7.7) by simply observing that, e.g., for $x > 0$,

$$E_{1,\alpha+1}(x) = \frac{x^{-\alpha}e^{x}}{\Gamma(\alpha+1)}\left(x^{\alpha}e^{-x} + \int_0^{x} u^{\alpha}e^{-u}\mathrm{d}u\right) = \frac{x^{-\alpha}e^{x}}{\Gamma(\alpha+1)}\left(\alpha\int_0^{x} u^{\alpha-1}e^{-u}\,\mathrm{d}u\right).$$

■

Lemma 7.4. *For $\alpha \in (0,2)$, the function $E_{1,\alpha}$ admits the following representation. For $x > 0$,*

$$E_{1,\alpha}(x) = x^{1-\alpha}\left(e^{x} + \frac{\sin(\alpha\pi)}{\pi}\int_0^{\infty} \frac{\xi^{1-\alpha}}{\xi+1}\, e^{-x\xi}\,\mathrm{d}\xi\right).$$

PROOF
Writing $\frac{e^{-\xi x}}{\xi+1} = e^{x}\int_x^{\infty} e^{-(\xi+1)u}\,\mathrm{d}u$, we obtain

$$\begin{aligned}\int_0^{\infty} \frac{\xi^{1-\alpha}}{\xi+1}\, e^{-x\xi}\,\mathrm{d}\xi &= e^{x}\int_0^{\infty} \xi^{1-\alpha}\,\mathrm{d}\xi \int_x^{\infty} e^{-(\xi+1)u}\,\mathrm{d}u = \Gamma(2-\alpha)e^{x}\int_x^{\infty} u^{\alpha-2}e^{-u}\,\mathrm{d}u \\ &= \frac{\Gamma(2-\alpha)}{1-\alpha}\, e^{x}\left(x^{\alpha-1}e^{-x} - \int_x^{\infty} u^{\alpha-1}e^{-u}\,\mathrm{d}u\right) \\ &= \Gamma(1-\alpha)\, e^{x}\left(x^{\alpha-1}e^{-x} - \Gamma(\alpha) + \int_0^{x} u^{\alpha-1}e^{-u}\,\mathrm{d}u\right).\end{aligned}$$

From this, it entails that

$$\int_0^x u^{\alpha-1}e^{-u}\,\mathrm{d}u = \frac{e^{-x}}{\Gamma(1-\alpha)}\int_0^\infty \frac{\xi^{1-\alpha}}{\xi+1}\,e^{-x\xi}\,\mathrm{d}\xi + \Gamma(\alpha) - x^{\alpha-1}e^{-x}$$

which, by (7.7), proves Lemma 7.4. ■

Lemma 7.5. *The Mittag-Leffler functions $E_{1,\frac{1}{2}}$ and $E_{\frac{1}{2},\frac{1}{2}}$ are related to the error function according to*

$$E_{1,\frac{1}{2}}(x) = \begin{cases} \dfrac{1}{\sqrt{\pi}} + \sqrt{x}\,e^x\mathrm{Erf}(\sqrt{x}\,) & \text{for } x \geq 0, \\ \dfrac{1}{\sqrt{\pi}} - \sqrt{|x|}\,e^x\mathrm{Erf}(\sqrt{|x|}\,) & \text{for } x \leq 0, \end{cases}$$

$$E_{\frac{1}{2},\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}} + xe^{x^2} + |x|e^{x^2}\mathrm{Erf}(|x|) \quad \text{for } x \in \mathbb{R}.$$

In particular,

$$E_{\frac{1}{2},\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}} + xe^{x^2}\mathrm{Erfc}(|x|) \quad \text{for } x \leq 0.$$

PROOF

Recall that $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_0^{x^2}\frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u$ and $\mathrm{Erfc}(x) = 1 - \mathrm{Erf}(x)$. By (7.7), we have, e.g., for $x \geq 0$, that

$$E_{1,\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}}\left(1 + \sqrt{x}\,e^x\int_0^x \frac{e^{-u}}{\sqrt{u}}\,\mathrm{d}u\right) = \frac{1}{\sqrt{\pi}} + \sqrt{x}\,e^x\mathrm{Erf}(\sqrt{x}).$$

On the other hand, we have, for $x \leq 0$,

$$E_{\frac{1}{2},\frac{1}{2}}(x) = \sum_{r=0}^\infty \frac{x^r}{\Gamma\left(\frac{r+1}{2}\right)} = \sum_{\ell=0}^\infty \frac{x^{2\ell}}{\Gamma\left(\ell+\frac{1}{2}\right)} + \sum_{\ell=0}^\infty \frac{x^{2\ell+1}}{\Gamma(\ell+1)} = E_{1,\frac{1}{2}}(x^2) + xe^{x^2}$$

from which we immediately extract the aforementioned representation. ■

Lemma 7.6. *For $\alpha \neq 0$, the following equality holds:*

$$\int_0^\infty \frac{e^{-w^2}}{w^2+\alpha^2}\,\mathrm{d}w = \frac{\pi}{2\alpha}\,e^{\alpha^2}\mathrm{Erfc}(\alpha).$$

PROOF

Put, for $\alpha \neq 0$,

$$F(\alpha) = \alpha\int_0^\infty \frac{e^{-w^2}}{w^2+\alpha^2}\,\mathrm{d}w = \int_0^\infty \frac{e^{-\alpha^2w^2}}{w^2+1}\,\mathrm{d}w.$$

We plainly have

$$\begin{aligned} F'(\alpha) &= -2\alpha\int_0^\infty \frac{w^2}{w^2+1}\,e^{-\alpha^2w^2}\,\mathrm{d}w = -2\alpha\int_0^\infty \left(1 - \frac{1}{w^2+1}\right)e^{-\alpha^2w^2}\,\mathrm{d}w \\ &= -2\alpha\int_0^\infty e^{-\alpha^2w^2}\,\mathrm{d}w + 2\alpha\,F(\alpha). \end{aligned}$$

So, the function F solves the differential equation $F'(\alpha) = 2\alpha F(\alpha) - \sqrt{\pi}$ with initial value $F(0) = \frac{\pi}{2}$. We deduce that $F(\alpha)$ has the form $F(\alpha) = G(\alpha)\,e^{\alpha^2}$ where the unknown function G satisfies $G'(\alpha) = -\sqrt{\pi}\,e^{-\alpha^2}$ and $G(0) = \frac{\pi}{2}$. This implies that $G(\alpha) = \sqrt{\pi}\int_\alpha^\infty e^{-\xi^2}\,\mathrm{d}\xi = \frac{\pi}{2}\,\mathrm{Erfc}(\alpha)$ and then $F(\alpha) = \frac{\pi}{2}\,e^{\alpha^2}\mathrm{Erfc}(\alpha)$. The proof of Lemma 7.6 is established. ■

Lemma 7.7. *The following identity holds: for $\lambda > 0$, $s > 0$ and $x > 0$,*

$$\int_0^s \frac{e^{-\lambda\sigma-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_{\sqrt{\lambda(s-\sigma)}}^\infty e^{-\xi^2}\,\mathrm{d}\xi\right)\mathrm{d}\sigma = \frac{\pi}{x}\,e^{\sqrt{\lambda}\,x}\,\mathrm{Erfc}\!\left(\frac{x}{2\sqrt{s}} + \sqrt{\lambda s}\right)$$

Proof

Put $I=\int_0^s \frac{e^{-\lambda\sigma-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_{\sqrt{\lambda(s-\sigma)}}^{\infty} e^{-\xi^2}\,\mathrm{d}\xi\right)\mathrm{d}\sigma$. With the change of variables $\xi=\sqrt{\lambda s z}$ and $v=z+\sigma/s$, we have

$$I=\frac{\sqrt{\lambda s}}{2}\int_0^s \frac{e^{-\lambda\sigma-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_{(s-\sigma)/s}^{\infty}\frac{e^{-\lambda s z}}{\sqrt{z}}\,\mathrm{d}z\right)\mathrm{d}\sigma=\frac{\sqrt{\lambda s}}{2}\int_0^s \frac{e^{-\frac{x^2}{4\sigma}}}{\sigma^{3/2}}\left(\int_1^{\infty}\frac{e^{-\lambda s v}}{\sqrt{v-\frac{\sigma}{s}}}\,\mathrm{d}v\right)\mathrm{d}\sigma.$$

With the new change of variable $u=s/\sigma$, it becomes

$$\begin{aligned}I&=\frac{\sqrt{\lambda s}}{2}\int_1^{\infty}\left(\frac{u}{s}\right)^{3/2}\frac{s}{u^2}\,e^{-\frac{x^2}{4s}u}\left(\int_1^{\infty}\frac{e^{-\lambda s v}}{\sqrt{v-\frac{1}{u}}}\,\mathrm{d}v\right)\mathrm{d}u=\frac{\sqrt{\lambda}}{2}\int_1^{\infty}\!\!\int_1^{\infty}\frac{e^{-\left(\frac{x^2}{4s}\right)u-\lambda s v}}{\sqrt{uv-1}}\,\mathrm{d}u\,\mathrm{d}v\\&=\frac{\sqrt{\lambda}}{2}\int_1^{\infty}\!\!\int_1^{\infty}\frac{e^{-\frac{a}{2}u-\frac{b}{2}v}}{\sqrt{uv-1}}\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

where we set $a=\frac{x^2}{2s}$ and $b=2\lambda s$. With the change of variable $(v,w)=(v,\frac{a}{2}u+\frac{b}{2}v)$, this gives

$$\begin{aligned}I&=\frac{\sqrt{\lambda}}{\sqrt{ab}}\int_{w=\frac{a+b}{2}}^{w=\infty}\int_{v=1}^{v=\frac{2w-a}{b}}\frac{e^{-w}}{\sqrt{\frac{2}{b}\,vw-v^2-\frac{a}{b}}}\,\mathrm{d}v\,\mathrm{d}w=\frac{1}{x}\int_{\frac{a+b}{2}}^{\infty}e^{-w}\left(\int_1^{\frac{2w-a}{b}}\frac{\mathrm{d}v}{\sqrt{\frac{w^2-ab}{b^2}-(v-\frac{w}{b})^2}}\right)\mathrm{d}w\\&=\frac{1}{x}\int_{\frac{a+b}{2}}^{\infty}\left(\arcsin\frac{w-a}{\sqrt{w^2-ab}}+\arcsin\frac{w-b}{\sqrt{w^2-ab}}\right)e^{-w}\,\mathrm{d}w.\end{aligned}$$

Let us integrate by parts this last integral. First, we have

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\arcsin\frac{w-a}{\sqrt{w^2-ab}}\right)=\frac{\sqrt{a}\,(w-b)}{(w^2-ab)\sqrt{2w-a-b}},$$

and then,

$$\begin{aligned}I&=\frac{1}{x}\left(\left[-\left(\arcsin\frac{w-a}{\sqrt{w^2-ab}}+\arcsin\frac{w-b}{\sqrt{w^2-ab}}\right)e^{-w}\right]_{\frac{a+b}{2}}^{\infty}+\int_{\frac{a+b}{2}}^{\infty}\frac{\sqrt{a}\,(w-b)+\sqrt{b}\,(w-a)}{(w^2-ab)\sqrt{2w-a-b}}\,e^{-w}\,\mathrm{d}w\right)\\&=\frac{\sqrt{a}+\sqrt{b}}{x}\int_{\frac{a+b}{2}}^{\infty}\frac{e^{-w}}{(w+\sqrt{ab})\sqrt{2w-(a+b)}}\,\mathrm{d}w=\frac{\sqrt{a}+\sqrt{b}}{x\sqrt{2}}\,e^{-\frac{a+b}{2}}\int_0^{\infty}\frac{e^{-w}}{(w+\frac{a+b}{2}+\sqrt{ab})\sqrt{w}}\,\mathrm{d}w\\&=\sqrt{2}\,\frac{\sqrt{a}+\sqrt{b}}{x}\,e^{-\frac{a+b}{2}}\int_0^{\infty}\frac{e^{-w^2}}{w^2+\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}}\right)^2}\,\mathrm{d}w.\end{aligned}$$

As a byproduct, in view of Lemma 7.6, we have

$$I=\sqrt{2}\,\frac{\sqrt{a}+\sqrt{b}}{x}\,\frac{\pi e^{-\frac{a+b}{2}}}{\sqrt{2}(\sqrt{a}+\sqrt{b})}\,e^{\frac{a+b}{2}+\sqrt{ab}}\,\mathrm{Erfc}\!\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}}\right)=\frac{\pi}{x}\,e^{\sqrt{\lambda}\,x}\,\mathrm{Erfc}\!\left(\frac{x}{2\sqrt{s}}+\sqrt{\lambda s}\right).$$

Lemma 7.7 is proved. ■

Lemma 7.8. *For any integer $m\le N-1$ and any $x\ge 0$,*

$$\int_0^{\infty}e^{-\lambda u}I_{j,m}(u;x)\,\mathrm{d}u=\lambda^{-\frac{m}{N}}e^{-\theta_j\sqrt[N]{\lambda}\,x}. \tag{2.16}$$

Proof

This formula is proved in [13] for $0\le m\le N-1$. To prove that it holds true also for negative m, we directly compute the Laplace transform of $I_{j,m}(u;x)$. We have

$$\int_0^{\infty}e^{-\lambda u}I_{j,m}(u;x)\,\mathrm{d}u=\frac{Ni}{2\pi}\left(e^{-i\frac{m}{N}\pi}\int_0^{\infty}\frac{\xi^{N-m-1}}{\xi^N+\lambda}\,e^{-\theta_j e^{i\frac{\pi}{N}}x\xi}\,\mathrm{d}\xi-e^{i\frac{m}{N}\pi}\int_0^{\infty}\frac{\xi^{N-m-1}}{\xi^N+\lambda}\,e^{-\theta_j e^{-i\frac{\pi}{N}}x\xi}\,\mathrm{d}\xi\right).$$

Let us integrate the function $H : z \to \frac{z^{M-1}}{z^N+\lambda}\, e^{-az}$ for fixed a and M such that $\Re(a) > 0$ and $M > 0$ on the contour $\Gamma_R = \{\rho e^{i\varphi} \in \mathbb{C} : \varphi = 0, \rho \in [0, R]\} \cup \{\rho e^{i\varphi} \in \mathbb{C} : \varphi \in (0, -\frac{2\pi}{N}), \rho = R\} \cup \{\rho e^{i\varphi} \in \mathbb{C} : \varphi = -\frac{2\pi}{N}, \rho \in (0, R]\}$. We get, by residues theorem,

$$\begin{aligned}
-\int_0^\infty \frac{z^{M-1}}{z^N+\lambda}\, e^{-az}\, \mathrm{d}z + e^{-2i\frac{M}{N}\pi} \int_0^\infty \frac{z^{M-1}}{z^N+\lambda}\, e^{-ae^{-i\frac{2\pi}{N}}z}\, \mathrm{d}z &= 2i\pi\, \mathrm{Residue}\Big(H, \sqrt[N]{\lambda}\, e^{-i\frac{\pi}{N}}\Big) \\
&= \frac{2i\pi}{N}\, \big(\sqrt[N]{\lambda}\big)^{M-N}\, e^{-i\frac{M-N}{N}\pi} e^{-a\sqrt[N]{\lambda}\, e^{-i\frac{\pi}{N}}} \\
&= \frac{2\pi}{Ni}\, \lambda^{\frac{M}{N}-1} e^{-i\frac{M}{N}\pi} e^{-a\sqrt[N]{\lambda}\, e^{-i\frac{\pi}{N}}}.
\end{aligned}$$

For $M = N - m$ and $a = \theta_j\, e^{i\frac{\pi}{N}} x$, this yields

$$\int_0^\infty e^{-\lambda u} I_{j;m}(u; x)\, \mathrm{d}u = -e^{-i\frac{m}{N}\pi} \lambda^{-\frac{m}{N}} \times (-e^{i\frac{m}{N}\pi}) e^{-\theta_j \sqrt[N]{\lambda}\, x} = \lambda^{-\frac{m}{N}} e^{-\theta_j \sqrt[N]{\lambda}\, x}.$$

Hence, (2.16) is valid for $m \le N - 1$. ■

Lemma 7.9. *The following identity holds: for any $x \in \mathbb{R}$ and $0 < s < t$,*

$$\begin{aligned}
&\int_0^\infty \xi^2 \sin(x\xi) e^{-t\xi^2} \left(\int_0^\xi e^{su^2}\, \mathrm{d}u \right) \mathrm{d}\xi \\
&\qquad = \frac{\sqrt{\pi}}{8} \frac{2t-s}{t^2(t-s)^{3/2}}\, x e^{-\frac{x^2}{4(t-s)}} - \frac{\pi}{8\sqrt{s}\, t^{3/2}} \left(1 - \frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}}\, \mathrm{Erf}\left(-\frac{1}{2}\sqrt{\frac{s}{t(t-s)}}\, x \right).
\end{aligned} \tag{7.11}$$

Proof
Set

$$\begin{aligned}
A &= \int_0^\infty \xi^2 \sin(x\xi) e^{-t\xi^2} \left(\int_0^\xi e^{su^2}\, \mathrm{d}u \right) \mathrm{d}\xi, \\
B &= \frac{\sqrt{\pi}}{8} \frac{2t-s}{t^2(t-s)^{3/2}}\, x e^{-\frac{x^2}{4(t-s)}} - \frac{\pi}{8\sqrt{s}\, t^{3/2}} \left(1 - \frac{x^2}{2t}\right) e^{-\frac{x^2}{4t}}\, \mathrm{Erf}\left(-\frac{1}{2}\sqrt{\frac{s}{t(t-s)}}\, x \right).
\end{aligned}$$

Using the expansion of the sine function, we get

$$A = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \int_0^\infty \left(\xi^{2n+3} e^{-t\xi^2} \right) \left(\int_0^\xi e^{su^2}\, \mathrm{d}u \right) \mathrm{d}\xi.$$

In order to integrate by parts this last integral, we search for a primitive of $\xi^{2n+3} e^{-t\xi^2}$. With the change of variable $\zeta = t\xi^2$, we have

$$\int \xi^{2n+3} e^{-t\xi^2}\, \mathrm{d}\xi = \frac{1}{2t^{n+2}} \int \zeta^{n+1} e^{-\zeta}\, \mathrm{d}\zeta = -\frac{(n+1)!}{2t^{n+2}} \left(\sum_{k=0}^{n+1} \frac{\zeta^k}{k!} \right) e^{-\zeta} = -\frac{(n+1)!}{2t^{n+2}} \sum_{k=0}^{n+1} \frac{t^k \xi^{2k}}{k!}\, e^{-t\xi^2}.$$

Then,

$$\begin{aligned}
A &= \frac{1}{2} \sum_{n=0}^\infty (-1)^n \frac{(n+1)!}{(2n+1)!\, t^{n+2}} \left[\sum_{k=0}^{n+1} \frac{t^k}{k!} \int_0^\infty \xi^{2k} e^{-t\xi^2}\, \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\int_0^\xi e^{su^2}\, \mathrm{d}u \right) \mathrm{d}\xi \right] x^{2n+1} \\
&= \frac{x}{4t^2\sqrt{t-s}} \sum_{n=0}^\infty (-1)^n \frac{(n+1)!}{(2n+1)!\, t^n} \left[\sum_{k=0}^{n+1} \frac{\Gamma\big(k+\frac{1}{2}\big)}{k!} \left(\frac{t}{t-s} \right)^k \right] x^{2n}.
\end{aligned} \tag{7.12}$$

On the other hand, by expanding $\mathrm{Erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \sum_{n=0}^\infty (-1)^n \frac{u^2}{n!}\, \mathrm{d}u = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \frac{\xi^{2n+1}}{(2n+1)n!}$, we get

$$\begin{aligned}
B &= \frac{\sqrt{\pi}}{8} \frac{2t-s}{t^2(t-s)^{3/2}}\, x e^{-\frac{x^2}{4(t-s)}} \\
&\quad - \frac{\pi}{8\sqrt{s}\, t^{3/2}} \left(\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{n!(4t)^n} - 2\sum_{n=0}^\infty \frac{(-1)^n x^{2n+2}}{n!(4t)^{n+1}} \right) \frac{2x}{\sqrt{\pi}} \sum_{n=0}^\infty \left(\frac{1}{2}\sqrt{\frac{s}{t(t-s)}} \right)^{2n+1} \frac{(-1)^{n+1} x^{2n}}{(2n+1)n!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\pi}}{8t^2}\frac{x}{\sqrt{t-s}}\left[\frac{2t-s}{t-s}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(4(t-s))^n}\right.\\
&\quad\left.+\left(\sum_{p=0}^{\infty}(-1)^p\frac{2p+1}{p!}\frac{x^{2p}}{(4t)^p}\right)\left(\sum_{q=0}^{\infty}\frac{(-1)^q}{(2q+1)q!}\left(\frac{s}{4t(t-s)}\right)^q x^{2q}\right)\right].
\end{aligned}
\tag{7.13}
$$

The computation of the above product of series can be carried out as follows:

$$
\begin{aligned}
&\left(\sum_{p=0}^{\infty}(-1)^p\frac{2p+1}{p!}\frac{x^{2p}}{(4t)^p}\right)\left(\sum_{q=0}^{\infty}\frac{(-1)^q}{(2q+1)q!}\left(\frac{s}{4t(t-s)}\right)^q x^{2q}\right)\\
&\qquad= \sum_{n=0}^{\infty}\left[\sum_{\substack{p,q\geq 0:\\ p+q=n}}\frac{2p+1}{(2q+1)p!q!}\left(\frac{s}{t-s}\right)^q\right]\frac{(-1)^n x^{2n}}{(4t)^n}\\
&\qquad= \sum_{n=0}^{\infty}\left[\sum_{q=0}^{n}\frac{2n-2q+1}{2q+1}\binom{n}{q}\left(\frac{s}{t-s}\right)^q\right]\frac{(-1)^n x^{2n}}{n!(4t)^n}\\
&\qquad= \sum_{n=0}^{\infty}\left[(2n+2)\sum_{q=0}^{n}\frac{\binom{n}{q}}{2q+1}\left(\frac{s}{t-s}\right)^q-\sum_{q=0}^{n}\binom{n}{q}\left(\frac{s}{t-s}\right)^q\right]\frac{(-1)^n x^{2n}}{n!(4t)^n}\\
&\qquad= \sum_{n=0}^{\infty}\left[(2n+2)\sum_{q=0}^{n}\frac{\binom{n}{q}}{2q+1}\left(\frac{s}{t-s}\right)^q\right]\frac{(-1)^n x^{2n}}{n!(4t)^n}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(4(t-s))^n}.
\end{aligned}
\tag{7.14}
$$

By inserting (7.14) into (7.13), we derive

$$
B = \frac{\sqrt{\pi}}{8t^2}\frac{x}{\sqrt{t-s}}\sum_{n=0}^{\infty}\left[\left(\frac{t}{t-s}\right)^{n+1}+\sum_{q=0}^{n}\frac{\binom{n}{q}}{2q+1}\left(\frac{s}{t-s}\right)^q\right]\frac{(-1)^n x^{2n}}{n!(4t)^n}.
\tag{7.15}
$$

Writing now $(\frac{s}{t-s})^q = (\frac{t}{t-s}-1)^q = \sum_{k=0}^{q}(-1)^{q-k}\binom{q}{k}(\frac{t}{t-s})^k$, we have

$$
\sum_{q=0}^{n}\frac{\binom{n}{q}}{2q+1}\left(\frac{s}{t-s}\right)^q = \sum_{k=0}^{n}(-1)^k\left(\sum_{q=k}^{n}(-1)^q\frac{\binom{q}{k}\binom{n}{q}}{2q+1}\right)\left(\frac{t}{t-s}\right)^k.
\tag{7.16}
$$

The sum with respect to q can be calculated as follows:

$$
\begin{aligned}
\sum_{q=k}^{n}(-1)^q\frac{\binom{q}{k}\binom{n}{q}}{2q+1} &= \binom{n}{k}\sum_{q=k}^{n}(-1)^q\frac{\binom{n-k}{q-k}}{2q+1} = (-1)^k\binom{n}{k}\sum_{q=0}^{n-k}(-1)^q\frac{\binom{n-k}{q}}{2q+2k+1}\\
&= (-1)^k\binom{n}{k}\int_0^1\sum_{q=0}^{n-k}(-1)^q\binom{n-k}{q}\xi^{2q+2k}\,\mathrm{d}\xi\\
&= (-1)^k\binom{n}{k}\int_0^1(1-\xi^2)^{n-k}\xi^{2k}\,\mathrm{d}\xi = (-1)^k\frac{1}{2}\binom{n}{k}\int_0^1(1-\xi)^{n-k}\xi^{k-1/2}\,\mathrm{d}\xi\\
&= (-1)^k\frac{1}{2}\binom{n}{k}B\left(n-k+1,k+\frac{1}{2}\right) = (-1)^k\frac{n!\,\Gamma\big(k+\frac{1}{2}\big)}{2k!\,\Gamma\big(n+\frac{3}{2}\big)}.
\end{aligned}
\tag{7.17}
$$

Plugging (7.17) into (7.16), and next (7.16) into (7.15), we obtain

$$
\begin{aligned}
B &= \frac{\sqrt{\pi}}{8t^2}\frac{x}{\sqrt{t-s}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(4t)^n}\left[\left(\frac{t}{t-s}\right)^{n+1}+\frac{(n+1)!}{\Gamma\big(n+\frac{3}{2}\big)}\sum_{k=0}^{n}\frac{\Gamma\big(k+\frac{1}{2}\big)}{k!}\left(\frac{t}{t-s}\right)^k\right]x^{2n}\\
&= \frac{\sqrt{\pi}}{8t^2}\frac{x}{\sqrt{t-s}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(4t)^n}\frac{(n+1)!}{\Gamma\big(n+\frac{3}{2}\big)}\left[\sum_{k=0}^{n+1}\frac{\Gamma\big(k+\frac{1}{2}\big)}{k!}\left(\frac{t}{t-s}\right)^k\right]x^{2n}\\
&= \frac{x}{4t^2\sqrt{t-s}}\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)!}{(2n+1)!t^n}\left[\sum_{k=0}^{n+1}\frac{\Gamma\big(k+\frac{1}{2}\big)}{k!}\left(\frac{t}{t-s}\right)^k\right]x^{2n}
\end{aligned}
\tag{7.18}
$$

where we used in the last equality $\Gamma\big(n+\frac{3}{2}\big)=\frac{\sqrt{\pi}}{2\cdot 4^n}\,\frac{(2n+1)!}{n!}$. In view of (7.12) and (7.18), we see that both members of (7.11) are equal: $A=B$. $\blacksquare$

Lemma 7.10. *We have, for $x\leq 0$,*

$$\int_0^{\infty} y(y-x)\,e^{-\frac{y^2}{4s}-\frac{(y-x)^2}{4(t-s)}}\,\mathrm{d}y$$
$$=-2\,\frac{s(t-s)^2}{t^2}\,xe^{-\frac{x^2}{4(t-s)}}+2\sqrt{\pi}\,\frac{s^{3/2}(t-s)^{3/2}}{t^{3/2}}\left(1-\frac{x^2}{2t}\right)e^{-\frac{x^2}{4t}}\,\mathrm{Erfc}\bigg(-\frac{1}{2}\sqrt{\frac{s}{t(t-s)}}\,x\bigg),$$

and, for $x\geq 0$,

$$\int_0^{\infty} y(y+x)\,e^{-\frac{(y+x)^2}{4s}-\frac{y^2}{4(t-s)}}\,\mathrm{d}y$$
$$=2\,\frac{s^2(t-s)}{t^2}\,xe^{-\frac{x^2}{4s}}+2\sqrt{\pi}\,\frac{s^{3/2}(t-s)^{3/2}}{t^{3/2}}\left(1-\frac{x^2}{2t}\right)e^{-\frac{x^2}{4t}}\,\mathrm{Erfc}\bigg(\frac{1}{2}\sqrt{\frac{t-s}{s\,t}}\,x\bigg).$$

Proof

We only produce the proof for $x\leq 0$ since the case $x\geq 0$ is quite similar. We have

$$\begin{aligned}
&\int_0^{\infty} y(y-x)\,e^{-\frac{y^2}{4s}-\frac{(y-x)^2}{4(t-s)}}\,\mathrm{d}y\\
&=e^{-\frac{x^2}{4t}}\int_0^{\infty} y(y-x)\,e^{-\frac{t}{4s(t-s)}(y-\frac{sx}{t})^2}\,\mathrm{d}y\\
&=e^{-\frac{x^2}{4t}}\int_{-sx/t}^{\infty}\left(y+\frac{sx}{t}\right)\left(y-\frac{(t-s)x}{t}\right)e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y\\
&=e^{-\frac{x^2}{4t}}\int_{-sx/t}^{\infty}\left(y^2-\frac{s(t-s)}{t^2}\,x^2\right)e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y+\frac{2s-t}{t}\,xe^{-\frac{x^2}{4t}}\int_{-sx/t}^{\infty} ye^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y\\
&=e^{-\frac{x^2}{4t}}\int_{-sx/t}^{\infty} y^2e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y-\frac{s(t-s)}{t^2}\,x^2e^{-\frac{x^2}{4t}}\int_{-sx/t}^{\infty} e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y+2\,\frac{s(t-s)(2s-t)}{t^2}\,xe^{-\frac{x^2}{4(t-s)}}.
\end{aligned}$$

Integrating by parts, we observe that

$$\int_{-sx/t}^{\infty} y^2e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y=2\,\frac{s^2(t-s)}{t^2}\,xe^{-\frac{sx^2}{4t(t-s)}}+2\,\frac{s(t-s)}{t}\int_{-sx/t}^{\infty} e^{-\frac{t}{4s(t-s)}y^2}\,\mathrm{d}y,$$

we finally get the result. $\blacksquare$

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